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Uniqueness and Clustering Properties of Gibbs States for Classical and Quantum Unbounded Spin Systems

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We consider quantum unbounded spin systems (lattice boson systems) in ν-dimensional lattice space $\mathbf{Z}^\nu$. Under appropriate conditions on the interactions we prove that in a region of high temperatures the Gibbs state is unique, is translationally invariant, and has clustering properties. The main methods we use are the Wiener integral representation, the cluster expansions for zero boundary conditions and for general Gibbs state, and explicitly β-dependent probability estimates. For one-dimensional systems we show the uniqueness of Gibbs states for any value of temperature by using the method of perturbed states. We also consider classical unbounded spin systems. We derive necessary estimates so that all of the results for the quantum systems hold for the classical systems by straightforward applications of the methods used in the quantum case.

KEY WORDS: Quantum unbounded spin systems; Wiener integral; Gibbs states; cluster expansion; clustering property; probability estimates.

1. INTRODUCTION

We continue our study of quantum unbounded spin systems (lattice boson systems) initiated in ref. 26. The model we consider can be viewed as a model for the quantum crystals[(5)] and is closely related to lattice field theory with continuous time.[(1)] In ref. 26 we gave a characterization of the Gibbs states in terms of conditional reduced density matrices and investigated the structure of the space of Gibbs states such as existence,

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convexity, and compactness of the space (see Section 2.1). In order to understand the phase transition phenomena, it may be necessary to show the uniqueness of the Gibbs states in a region of high temperatures, and then the nonuniqueness of the states at low temperatures.[3,11,27] In this paper we prove the uniqueness and the clustering properties of the Gibbs states in a region of high temperatures under appropriate conditions on interactions (two-body interactions). In the case of one-dimensional systems the uniqueness holds for any value of temperature. See Section 2.2 for the details. The methods for the quantum systems can be applied straightforwardly to the classical systems to show that all of the results for the quantum systems also hold for the classical systems. We plan to investigate the detailed structure of the phase diagram in the near future.

The main methods we use are the Wiener integral representation[13,33] and cluster expansions for zero boundary conditions and for general Gibbs states.[23] For classical systems a cluster expansion has been developed in terms of polymer systems[17] and the analyticities and the clustering properties of correlation functions have been established. For quantum systems one of us gave a sketch of a cluster expansion for zero boundary conditions.[25] However, the uniqueness of Gibbs states for classical as well as quantum systems remains open. See Conjecture 4.1 of ref. 26. In this paper we develop a cluster expansion method for quantum systems by using the Wiener integral representation and modifying the cluster expansion method developed in refs. 23 and 25 to show the uniqueness of Gibbs states in a region of high temperatures. In ref. 22 one-dimensional systems are studied by means of a cluster expansion of polymer types and an infinite-volume limiting Gibbs state which is translationally invariant and ergodic is constructed. By using a method of perturbed states[2,15] we prove that the state constructed in ref. 22 is the unique Gibbs state for one-dimensional systems.

Let us describe the main mthods in this paper briefly. As in ref. 26, we use the Wiener integral representation.[13,33] For $\beta > 0$, let S^β be the space of continuous loops from $[0, \beta]$ to $\mathbf{R}^d$, i.e., $S^\beta = \{s\colon [0, \beta] \to \mathbf{R}^d\colon s(0) = s(\beta)\}$. We give an *a priori* measure λ_β by $\lambda_\beta(ds) = dx\ P^\beta(d\omega)$, $s = x + \omega$, where dx is the Lebesgue measure on $\mathbf{R}^d$ and $P^\beta(d\omega)$ is the conditional Wiener measure.[33] For a finite set $\Lambda \subset \mathbf{Z}^\nu$ we use the following notation:

$$x_\Lambda = \{x_i : i \in \Lambda\}, \qquad dx_\Lambda = \prod_{i \in \Lambda} dx_i$$
$$s_\Lambda = \{s_i : s_i \in S^\beta, i \in \Lambda\}, \qquad \lambda_\beta(ds_\Lambda) = \prod_{i \in \Lambda} \lambda_\beta(ds_i) \tag{1.1}$$

and $\Omega_\Lambda^\beta = (S^\beta)^\Lambda$. Put $\Omega^\beta = (S^\beta)^{\mathbf{Z}^\nu}$. We consider the following type of interaction (two-body interactions):

$$V(x_\Lambda) = \sum_{i \in \Lambda} P(x_i) + \sum_{\{i,j\} \subset \Lambda} U(x_i, x_j; |i-j|) \tag{1.2}$$

where P and U are the self-interaction and the two-body interaction, respectively, which satisfy specific conditions (Assumption 2.2.1). Put

$$W(x_\Lambda, x_{\Lambda^c}) = \sum_{\substack{\{i,j\}: \\ i \in \Lambda, j \in \Lambda^c}} U(x_i, x_j; |i-j|) \tag{1.3}$$

We write

$$V_\beta(s_\Lambda) = \int_0^\beta V(s_\Lambda(\tau))\, d\tau \tag{1.4}$$

and $W_\beta(s_\Lambda, s_{\Lambda^c})$ analogously. For $\Lambda \subset \mathbf{Z}^\nu$ we denote by $\mathscr{F}_\Lambda$ the local σ-algebra on Ω^β and put $\mathscr{F} = \mathscr{F}_{\mathbf{Z}^\nu}$. See Section 2 for the details.

For finite $\Lambda \subset \mathbf{Z}^\nu$, let $\nu_{\Lambda,\beta}^{(0)}$ be the local Gibbs measure on Ω_Λ^β with zero boundary conditions and let ν_β be a Gibbs measure on Ω^β. See Section 2.1 for the definitions. In Section 3 we develop the cluster expansions of the following types: For any $\Delta \subset \Lambda \subset \mathbf{Z}^\nu$ and any $\mathscr{F}_\Delta$-measurable bounded function f

$$\nu_{\Lambda,\beta}^{(0)}(f) = \sum_{\phi \subseteq X \subset \Lambda} K_\beta(\Delta, X; f)\, g_{\Lambda,\beta}(\Delta \cup X) \tag{1.5}$$

and

$$\nu_\beta(f) = \sum_{\phi \subseteq X \subset \Lambda} K_\beta(\Delta, X; f)\, \tilde{g}_\beta(\Delta \cup X) + R_{\Lambda,\beta}(f) \tag{1.6}$$

We then show that

$$\lim_{\Lambda \to \mathbf{Z}^\nu} R_{\Lambda,\beta}(f) = 0$$

By using the method of Kirkwood–Salsburg type integral equations[30] we prove that for any $X \subset \mathbf{Z}^\nu$

$$\lim_{\Lambda \to \mathbf{Z}^\nu} g_{\Lambda,\beta}(X) = \tilde{g}_\beta(X)$$

Thus we conclude that for any Gibbs state ν_β

$$\nu_\beta(f) = \lim_{\Lambda \to \mathbf{Z}^\nu} \nu^{(0)}_{\Lambda,\beta}(f)$$

and so the uniqueness of Gibbs measures follows from the above. This result and the definition of Gibbs states (Definition 2.1.4) imply the uniqueness of Gibbs states. The cluster properties of the unique Gibbs state follow from a consequence of the convergence of the cluster expansion (see Section 6.3).

For one-dimensional systems we use the method of perturbed states.[2,15,29] Let ν_β be a fixed extremal Gibbs measure which is translationally invariant. For any interval $\Lambda = [-n, n]$ it turns out that $\exp[W_\beta(s_\Lambda, s_{\Lambda^c})]$ is an element in $L^2(\Omega^\beta, d\nu_\beta)$. By the equilibrium conditions we have that for any $\mathscr{F}_\Lambda$-measurable bounded function f

$$\nu^{(0)}_{\Lambda,\beta}(f) = \int \nu_\beta(ds) f(s) \exp[W_\beta(s_\Lambda, s_{\Lambda^c})]/\tilde{N}_{\Lambda,\beta}$$

where $\tilde{N}_{\Lambda,\beta}$ is the normalization factor. By taking $n \to \infty$ and using the extremality of ν_β we show that for any $\beta > 0$

$$\nu_\beta(f) = \lim_{n \to \infty} \nu^{(0)}_{\Lambda_n,\beta}(f)$$

where $\Lambda_n = [-n, n]$. This implies the uniqueness. See Section 7 for the details.

In order to obtain the results in Section 2.2 for classical systems, one only needs to replace

$$S^\beta, \quad \lambda_\beta(ds), \quad \Omega^\beta, \quad V_\beta, \quad W_\beta$$

by

$$\mathbf{R}^d, \quad dx, \quad \Omega = (\mathbf{R}^d)^{\mathbf{Z}^\nu}, \quad \beta V, \quad \beta W$$

respectively. Then straightforward applications of the methods used for quantum systems give the results for classical systems. We supply the necessary estimates for classical systems in Section 4.2.

We organize the paper as follows: In Section 2.1 we introduce notations, definitions, and necessary preliminaries. In Section 2.2 we give basic assumptions on interactions and then list the main results in this paper. We develop a cluster expansion for zero boundary conditions in Section 3.1 and then derive a cluster expansion for general Gibbs measures in Section 3.2 by using the equilibrium conditions. Section 4 is devoted to *a priori* estimates. We also produce basic estimates for classical systems in Section 4.2. In Section 5 we prove the convergence of the cluster expansion for zero boundary conditions by using the basic estimates. In Section 6 we prove uniqueness and clustering properties in a region of high temperatures. The main tool is the method of Kirkwood–Salsburg type integral equations.[30] The proof of the uniqueness of Gibbs states for one-dimensional systems for any values of $\beta > 0$ is given in Section 7. In the Appendix we produce the proofs of explicitly β-dependent probability estimates (Propositions 4.1.3 and 4.2.3).

2. PRELIMINARIES AND MAIN RESULTS

2.1. Preliminaries

We consider quantum unbounded spin systems (lattice boson systems) on the ν-dimensional lattice space $\mathbf{Z}^\nu$. We collect notations, definitions, and some results from ref. 26 which will be used in the sequel. By $\mathscr{C}$ we mean the class of finite subsets of $\mathbf{Z}^\nu$. At each site $i \in \mathbf{Z}^\nu$ we associate an identical copy of the Hilbert space $L^2(\mathbf{R}^d, dx)$, where dx is the Lebesgue measure on $\mathbf{R}^d$. For $x = (x^1,..., x^d) \in \mathbf{R}^d$ and $i = (i_1,..., i_\nu) \in \mathbf{Z}^\nu$ we write

$$|x| = \left[\sum_{l=1}^{d} (x^l)^2 \right]^{1/2}, \qquad |i| = \max_{1 \leqslant l \leqslant \nu} |i_l| \tag{2.1}$$

For any bounded region $\Lambda \subset \mathbf{Z}^\nu$ we write

$$x_\Lambda = \{x_i : i \in \Lambda\}, \qquad dx_\Lambda = \prod_{i \in \Lambda} dx_i \tag{2.2}$$

The (local) Hilbert space for lattice boson systems in $\Lambda \in \mathscr{C}$ is given by

$$\begin{aligned} \mathfrak{H}_\Lambda &= \bigotimes_{i \in \Lambda} L^2(\mathbf{R}^d, dx_i) \\ &= L^2((\mathbf{R}^d)^\Lambda, dx_\Lambda) \end{aligned} \tag{2.3}$$

and a (local) Hamiltonian operator on $\mathfrak{H}_\Lambda$ is given by

$$\begin{aligned} H_\Lambda &= -\tfrac{1}{2} \sum_{i \in \Lambda} \Delta_i + V(x_\Lambda) \\ V(x_\Lambda) &\equiv \sum_{\Delta \subset \Lambda} \Phi_\Delta(x_\Delta) \end{aligned} \tag{2.4}$$

where Δ_i is the Laplacian operator for the variable $x_i \in \mathbf{R}^d$ and for each $\Delta \subset \mathbf{Z}^\nu$, and Φ_Δ is the interaction potential, which is a measurable real-valued function on $(\mathbf{R}^d)^\Delta$.

As in ref. 26, we impose the following conditions on the potential Φ:

Assumption 2.1.1. The potential $\Phi = (\Phi_\Delta)_{\Delta \subset \mathbf{Z}^\nu}$ satisfies the following conditions:

(a) Φ_Δ is a Borel measurable function on $(\mathbf{R}^d)^\Delta$.

(b) Φ_Δ is invariant under translations on $\mathbf{Z}^\nu$.

(c) (Superstability) There are $A > 0$ and $c \in \mathbf{R}$ such that

$$V(x_\Lambda) = \sum_{\Delta \subset \Lambda} \Phi_\Delta(x_\Delta) \geqslant \sum_{i \in \Lambda} (Ax_i^2 - c)$$

(d) (Strong regularity) There exists a decreasing positive function Ψ on the natural integers such that

$$\Psi(r) \leqslant Kr^{-\nu-\varepsilon} \qquad \text{for some} \quad K \text{ and } \varepsilon > 0 \text{ with } \sum_{i \in \mathbf{Z}^\nu} \Psi(|i|) < A$$

Furthermore if Λ_1 and Λ_2 are disjoint finite subsets of $\mathbf{Z}^\nu$ and if one writes

$$V(x_{\Lambda_1 \cup \Lambda_2}) = V(x_{\Lambda_1}) + V(x_{\Lambda_2}) + W(x_{\Lambda_1}, x_{\Lambda_2})$$

then the bound

$$|W(x_{\Lambda_1}, x_{\Lambda_2})| \leqslant \sum_{i \in \Lambda_1} \sum_{j \in \Lambda_2} \Psi(|i-j|) \tfrac{1}{2}(x_i^2 + x_j^2)$$

holds.

For a bounded domain $\Lambda \subset \mathbf{Z}^\nu$, the C^*-algebra of local observables is defined by

$$\mathfrak{A}_\Lambda = \mathfrak{L}(\mathfrak{H}_\Lambda) \tag{2.5}$$

where $\mathfrak{L}(\mathfrak{H}_\Lambda)$ is the algebra of all bounded operators on $\mathfrak{H}_\Lambda$. If $\Lambda_1 \cap \Lambda_2 = \varnothing$, then $\mathfrak{A}_{\Lambda_1 \cup \Lambda_2} = \mathfrak{A}_{\Lambda_1} \otimes \mathfrak{A}_{\Lambda_2}$, and $\mathfrak{A}_{\Lambda_1}$ is isomorphic to the C^*-algebra $\mathfrak{A}_{\Lambda_1} \otimes \mathbf{1}_{\Lambda_2}$, where $\mathbf{1}_{\Lambda_2}$ denotes the identity operator on $\mathfrak{H}_{\Lambda_2}$. In this way we identity $\mathfrak{A}_\Lambda$ as a subalgebra of $\mathfrak{A}_{\Lambda'}$ if $\Lambda \subset \Lambda'$. The quasilocal algebra of local observables is given by

$$\mathfrak{A} = \left(\bigcup_{\Lambda \in \mathscr{C}} \mathfrak{A}_\Lambda \right)^- \tag{2.6}$$

where the bar means the completion with respect to the uniform norm. Notice that $\mathfrak{A}$ is a unital C^*-algebra.

We next describe the Wiener integral formalism of lattice boson systems.(4,13,26,33) For x, $y \in \mathbf{R}^d$ and $\beta > 0$ let us denote by $W^\beta_{x,y}$ the set of continuous paths ω: $[0, \beta] \to \mathbf{R}^d$ with $\omega(0) = x$, $\omega(\beta) = y$. The set $W^\beta_{x,y}$ is endowed with the standard Borel space structure. For $\beta > 0$ denote by $P^\beta_{x,y}$ the conditional Wiener measure on $W_{x,y}$,(33)

$$P^\beta_{x,y}(W_{x,y}) = (2\pi\beta)^{-d/2} \exp\left(-\frac{1}{2\beta}|x-y|^2\right)$$

For finite $\Lambda \in \mathbf{Z}^\nu$, x_Λ, $y_\Lambda \in (\mathbf{R}^d)$, and $\beta > 0$, we use the notation

$$W^\beta_{x_\Lambda, y_\Lambda} = \mathop{\times}_{i \in \Lambda} W^\beta_{x_i, y_i} \qquad \text{and} \qquad P^\beta_{x_\Lambda, y_\Lambda} = \mathop{\times}_{i \in \Lambda} P^\beta_{x_i, y_i}$$

We identify the space $W^\beta_{x,x}$, $x \in \mathbf{R}^d$, with a single space $W^\beta = W^\beta_{0,0}$ by means of the mapping $\omega \leftrightarrow \omega + x$. The measures $P^\beta_{x,x}$ and $P^\beta = P^\beta_{0,0}$ are transformed thereby into each other. Furthermore, we use the map $W^\beta_{x,y} \leftrightarrow W^\beta_{0,0}$ given by $\omega \leftrightarrow \omega + L^\beta_{x,y}$ where $L^\beta_{x,y}$ is the linear function $L^\beta_{x,y}(t) = x + \beta^{-1}t(y-x)$. The measure $P^\beta_{x,y}$ is transformed thereby into $\exp[-(1/2\beta)\,|x-y|^2]\,P^\beta$. The product space $W^\beta_{x_\Lambda, y_\Lambda}$ is transformed into $(W^\beta_{0,0})^\Lambda$ analogously in which the function $L^\beta_{x_\Lambda, y_\Lambda}(t) = x_\Lambda + \beta^{-1}t(y_\Lambda - x_\Lambda)$ is used. We shall use the notation

$$S^\beta = \mathbf{R}^d \times W^\beta \qquad \text{and} \qquad s = (x, \omega) \in S^\beta \tag{2.7}$$

as well as

$$\hat{S}^\beta = \mathbf{R}^d \times \mathbf{R}^d \times W^\beta \qquad \text{and} \qquad \hat{s} = (x, y, \omega) \in \hat{S}^\beta \tag{2.8}$$

where $s(t) = x + \omega(t)$ and $\hat{s}(t) = \omega(t) + L^\beta_{x,y}(t)$, respectively. The set S^β has a σ-algebra generated by the products of Borel sets in $\mathbf{R}^d$ and cylinder sets in $W^\beta = W^\beta_{0,0}$.(33) We give an *a priori* measure λ_β on S^β by

$$\begin{aligned} &\lambda_\beta(ds) \equiv dx\, P^\beta(d\omega), \qquad s = (x, \omega) \in S^\beta \\ &\lambda_\beta(ds_\Lambda) = \prod_{i \in \Lambda} \lambda_\beta(ds_i) \end{aligned} \tag{2.9}$$

We use the notations

$$\Omega^\beta = (S^\beta)^{\mathbf{Z}^\nu} = (\mathbf{R}^d \times W^\beta)^{\mathbf{Z}^\nu} \tag{2.10}$$

and

$$\hat{\Omega}^\beta = (\hat{S}^\beta)^{\mathbf{Z}^\nu} = (\mathbf{R}^d \times \mathbf{R}^d \times W^\beta)^{\mathbf{Z}^\nu} \tag{2.11}$$

For each $i \in \mathbf{Z}^\nu$, let $P_i: \Omega^\beta \to S^\beta$ be the projection $P_i(s) = s_i$, the value (path) on the ith site. For each $\Lambda \subset \mathbf{Z}^\nu$, we have a local σ-algebra $\mathscr{F}_\Lambda$ which is the minimal σ-algebra for which P_i, $i \in \Lambda$, are measurable. We simply write $\mathscr{F}$ for $\mathscr{F}_{\mathbf{Z}^\nu}$. We write $\mathscr{P}(\Omega^\beta, \mathscr{F})$ for the family of probability measures on Ω^β.

We write that for $\Lambda \subset \mathbf{Z}^\nu$ and $s \in \Omega^\beta$

$$\begin{aligned} \Phi_{\Delta,\beta}(s_\Delta) &= \int_0^\beta \Phi_\Delta(s_\Delta(t))\, dt \\ V_\beta(s_\Lambda) &= \sum_{\Delta \subset \Lambda} \Phi_\Delta(s_\Delta) \\ W_\beta(s_\Lambda, s_{\Lambda^c}) &= \sum_{\Delta \cap \Lambda \neq \varnothing,\, \Delta \cap \Lambda^c \neq \varnothing} \Phi_\Delta(s_\Delta) \end{aligned} \tag{2.12}$$

Denote

$$\begin{aligned} s_i^2 &= \int_0^\beta s_i^2(t)\, dt, \qquad i \in \mathbf{Z}^\nu \\ \mathfrak{S}_N &= \left\{ s \in \Omega : \forall l, \sum_{|i| \leqslant l} s_i^2 \leqslant N^2(2l+1)^\nu \right\} \\ \mathfrak{S} &= \bigcup_{N \in \mathbf{N}} \mathfrak{S}_N \end{aligned} \tag{2.13}$$

We say that a measure μ on $(\Omega^\beta, \mathscr{F})$ is *tempered* if it has its support on $\mathfrak{S}$.[31] A Borel probability measure μ on $(\Omega^\beta, \mathscr{F})$ is said to be *regular* if there exist $\bar{A} > 0$ and $\bar{\delta}$ so that the projection $\mu(ds_\Lambda)$ of μ on any $(\Omega^\beta, \mathscr{F}_\Lambda)$ satisfies

$$g(s_\Lambda \mid \mu) \leqslant \exp\left[-\sum_{i \in \Lambda} (\bar{A} s_i^2 - \bar{\delta}) \right] \tag{2.14}$$

where $g(s_\Lambda \mid \mu)$ is such that $\mu(ds_\Lambda) = g(s_\Lambda \mid \mu)\, \lambda_\beta(ds_\Lambda)$. It is easy to check that any regular measure is tempered.[31]

Before listing the main results in ref. 26, it may be worthwhile to give a brief discussion on the main idea used to characterize the Gibbs states.[26] For $\Lambda \in \mathscr{C}$, let H_Λ be the local Hamiltonian given by (2.4). By the Feynman–Kac formula the operator $\exp(-\beta H_\Lambda)$ has its integral kernel[26,33]

$$e^{(-\beta H_\Lambda)}(x_\Lambda, y_\Lambda) = \int P_{x_\Lambda, y_\Lambda}(ds_\Lambda) \exp[-V_\beta(s_\Lambda)]$$

and so the finite-volume partition function for the local Hamiltonian H_Λ and the inverse temperature β is given by

$$\begin{aligned} Z^{\Phi}_{\Lambda,\beta} &\equiv \mathrm{Tr}_{\mathfrak{H}_\Lambda}[\exp(-\beta H_\Lambda)] \\ &= \int \lambda_\beta(ds_\Lambda)\exp[-V_\beta(s_\Lambda)] \end{aligned}$$

Similarly the local Gibbs states can be expressed by integrations on the path space $(\hat{S}^\beta)^\Lambda$. Employing standard methods in classical statistical mechanics,[12,19,28] we were able to introduce a family of conditional measures (specifications) and Gibbs measures on Ω^β. See (2.16) and Definition 2.1.2 stated below. The Gibbs states have been defined by the conditional reduced density matrices (2.18) and Gibbs measures. For the details we refer the reader to ref. 26. See also the discussion below.

Finally we collect definitions and basic results from ref. 26. The partition function in $\Lambda \in \mathscr{C}$ for the interaction Φ with boundary condition $\bar{s} \in \mathfrak{S}$ is given by

$$Z^{\Phi}_{\Lambda,\beta}(\bar{s}) = \int \lambda_\beta(ds_\Lambda)\exp[-V_\beta(s_\Lambda) - W_\beta(s_\Lambda, \bar{s}_{\Lambda^c})] \tag{2.15}$$

Notice that $Z^{\Phi}_{\Lambda,\beta}$ corresponds to the partition function with the zero boundary condition, i.e., $Z^{\Phi}_{\Lambda,\beta} = Z^{\Phi}_{\Lambda,\beta}(0)$. The Gibbs specification $\gamma^\Phi = (\gamma^\Phi_\Lambda)_{\Lambda\in\mathscr{C}}$ with respect to $\mathfrak{S}$ is defined by[12,26,28]

$$\gamma^\Phi_\Lambda(A \mid \bar{s}) = \begin{cases} Z^\Phi_\Lambda(\bar{s})^{-1} \int \lambda_\Lambda(ds_\Lambda)\exp[-V_\beta(s_\Lambda) - W_\beta(s_\Lambda, \bar{s}_{\Lambda^c})] \\ \quad \times 1_A(s_\Lambda\, \bar{s}_{\Lambda^c}) \qquad \text{if} \quad \bar{s} \in \mathfrak{S} \\ 0 \qquad \text{if} \quad \bar{s} \notin \mathfrak{S} \end{cases} \tag{2.16}$$

where $A \in \mathscr{F}$ and 1_A is the indicator function on A, and $s_\Lambda \bar{s}_{\Lambda^c}$ is the configuration defined by s_Λ on Λ and $\bar{s}_{\Lambda^c}$ on Λ^c, respectively. It can be checked that the Gibbs specification satisfies the consistency condition[12]: For $\Delta \subset \Lambda$, $\bar{s} \in \mathfrak{S}$,

$$\begin{aligned} \gamma^\Phi_\Lambda \gamma^\Phi_\Delta(A \mid \bar{s}) &\equiv \int_{\mathfrak{S}} \gamma^\Phi_\Lambda(ds^* \mid \bar{s})\, \gamma^\Phi_\Delta(A \mid s^*) \\ &= \gamma^\Phi_\Lambda(A \mid \bar{s}) \end{aligned} \tag{2.17}$$

The Gibbs measure on $(\Omega^\beta, \mathscr{F})$ is defined as follows[12,26]:

Definition 2.1.2. A Gibbs measure μ for the potential Φ is a Borel probability measure on $(\Omega^\beta, \mathscr{F})$ satisfying the equilibrium condition

$$\mu(A) = \int \mu(d\bar{s})\, \gamma_\Lambda^\Phi(A \mid \bar{s}), \qquad A \in \mathscr{F}$$

We denote by $\mathscr{G}^\Phi(\Omega^\beta)$ the family of all Gibbs measures on $(\Omega^\beta, \mathscr{F})$ for the interaction potential Φ.

We then have the following result:

Theorem 2.1.3 (Ref. 26, Theorem 2.7). Under Assumption 2.1.1 any Gibbs measure $\nu \in \mathscr{G}^\Phi(\Omega^\beta)$ is regular. Furthermore, $\mathscr{G}^\Phi(\Omega^\beta)$ is nonempty, convex, compact in the local convergence topology, and a Choquet simplex.

Let us now consider Gibbs (equilibrium) states on the quasilocal algebra $\mathfrak{A}$. For $\Lambda \in \mathscr{C}$ and a configuration $\bar{s} \in \mathfrak{S}$, we define a function $k_\Lambda(x_\Lambda, y_\Lambda; \bar{s})$, x_Λ, $y_\Lambda \in (\mathbf{R}^d)^\Lambda$, which takes the role of the conditional reduced density matrix[4,7-9]:

$$k_\Lambda(x_\Lambda, y_\Lambda; \bar{s}) = Z_{\Lambda,\beta}(\bar{s})^{-1} \int P^\beta_{x_\Lambda, y_\Lambda}(ds_\Lambda) \exp[-V_\beta(s_\Lambda) - W_\beta(s_\Lambda, \bar{s}_{\Lambda^c})] \tag{2.18}$$

With the help of these functions and the Gibbs measures we define the Gibbs (equilibrium) states as follows.[26]

Definition 2.1.4. A state ρ on the quasilocal algebra $\mathfrak{A}$ is called a Gibbs state if there exists a Gibbs measure $\nu \in \mathscr{G}^\Phi(\Omega^\beta)$ such that the restriction ρ_Λ of ρ to $\mathfrak{A}_\Lambda$ is given by

$$\rho_\Lambda(A) = \mathrm{Tr}_{\mathfrak{H}_\Lambda}(K_\Lambda^{(\nu)} A), \qquad A \in \mathfrak{A}_\Lambda$$

where the density matrix $K_\Lambda^{(\nu)}$ is defined by its integral kernel

$$K_\Lambda^{(\nu)}(x_\Lambda, y_\Lambda) = \int \nu(d\bar{s})\, k_\Lambda(x_\Lambda, y_\Lambda; \bar{s})$$

We denote by $\mathscr{G}_\beta^\Phi(\mathfrak{A})$ the family of Gibbs states for Φ on $\mathfrak{A}$ at inverse temperature $\beta > 0$.

Theorem 2.1.5 (Ref. 26, Theorem 2.9). Under Assumption 2.1.1, $\mathscr{G}_\beta^\Phi(\mathfrak{A})$ is nonempty, convex, and also weak*-compact if the interaction is of finite range.

2.2. Main Results

As stated in the Introduction, the purpose of this paper is to prove the uniqueness and the cluster properties of Gibbs states in a region of high temperature (sufficiently small $\beta > 0$). In order to avoid unnecessary complications, we make the following assumptions on the interaction potentials.

Assumption 2.2.1. The potential $\Phi = (\Phi_\Delta)_{\Delta \subset \mathbf{Z}^\nu}$ satisfies the following conditions:

(a) There exist Borel functions P and U on $\mathbf{R}^d$ and $\mathbf{R}^d \times \mathbf{R}^d \times \mathbf{N}$ such that

$$\Phi_{\{i\}}(x_i) = P(x_i), \qquad \Phi_{\{i,j\}}(x_i, x_j) = U(x_i, x_j; |i-j|)$$
$$\Phi_\Delta(x_\Delta) = 0 \qquad \text{if} \quad |\Delta| > 2$$

(b) There exist $\gamma > 2$, $D_1 > 0$, $D_2 > 0$, and $D_3 > 0$ such that the inequalities

$$D_1(|x|^\gamma - D_2) \leqslant P(x) \leqslant D_3(|x|^\gamma + 1), \qquad x \in \mathbf{R}^d$$

hold.

(c) There exists a decreasing positive function Ψ on the natural numbers such that

$$\Psi(r) \leqslant K r^{-\nu-\varepsilon} \qquad \text{for some} \quad K \text{ and } \varepsilon > 0$$

Furthermore, the bound

$$|U(|x_i - x_j|)| \leqslant \Psi(|i-j|)\, |x_i| \cdot |x_j|$$

holds. Here (and hereafter) we have used the notation $U(|x_i - x_j|) \equiv U(x_i, x_j; |i-j|)$.

Remark. Assumption 2.2.1(c) implies that for any $\Lambda \subset \mathbf{Z}^\nu$ the bound

$$\left| \sum_{\{i,j\} \subset \Lambda} U(|x_i - x_j|) \right| \leqslant J \sum_{i \in \Lambda} |x_i|^2 \tag{2.19}$$

holds for some $J > 0$. Thus from Assumption 2.2.1(a)–(b) and the above bound one has that

$$V(x_\Lambda) \geqslant \sum_{i \in \Lambda} [A' |x|^\gamma - \delta'] \tag{2.20}$$

for some positive constants $A' > 0$ and $\delta' > 0$, and so all conditions in Assumption 2.1.1 are satisfied.

The following are the main results in this paper:

Theorem 2.2.2 (Uniqueness of Gibbs states). Under Assumption 2.2.1 there exists $\beta_0 > 0$ such that for any β with $0 < \beta < \beta_0$ the Gibbs measure on $(\Omega^\beta, \mathscr{F})$ is unique, i.e., $\text{card}(\mathscr{G}^\Phi(\Omega^\beta)) = 1$. Consequently the Gibbs state on $\mathfrak{A}$ is unique.

Theorem 2.2.3 (Cluster property). Under Assumption 2.2.1, let $\rho \in \mathscr{G}_\beta^\Phi(\mathfrak{A})$ be the unique Gibbs state for $\beta < \beta_0$. Then for any $A \in \mathfrak{A}_\Lambda$, $B \in \mathfrak{A}_{\Lambda'}$

$$|\rho(AB) - \rho(A)\,\rho(B)| \to 0 \qquad \text{as} \quad \text{dist}(\Lambda, \Lambda') \to \infty$$

where $\text{dist}(\Lambda, \Lambda') = \inf\{|i-j| : i \in \Lambda, j \in \Lambda'\}$.

We next consider the quantum unbounded spin systems in the one-dimensional lattice space **Z**. Let Ψ be the function on **N** given in Assumption 2.2.1(c).

Theorem 2.2.4. Assume that $[\Psi(r)]^{1/2} \leqslant K r^{-1-\varepsilon}$ for some constants $K > 0$ and $\varepsilon > 0$. Then for any $\beta > 0$ there exists a unique translationally invariant Gibbs state.

The rest of the paper is devoted to the proof of Theorem 2.2.2–2.2.4.

3. CLUSTER EXPANSIONS

3.1. Cluster Expansion for Zero Boundary Conditions

We develop a cluster expansion for zero boundary conditions in this subsection. A cluster expansion for general Gibbs states will be developed in the following subsection. The methods we use are closely related to those in refs. 23 and 25. Using the notation $U(|x_i - x_j|) \equiv U(x_i, x_j; |i-j|)$ again, we write that for $\Lambda \in \mathscr{C}$

$$
\begin{aligned}
P_\beta(s_i) &= \int_0^\beta P(s_i(\tau))\, d\tau, \qquad P_\beta(s_\Lambda) = \sum_{i \in \Lambda} P_\beta(s_i) \\
U_\beta(|s_i - s_j|) &= \int_0^\beta U(|s_i(\tau) - s_j(\tau)|)\, d\tau, \\
U_\beta(s_\Lambda) &= \sum_{\{i,j\} \subset \Lambda} U_\beta(|s_i - s_j|) \\
V_\beta(s_\Lambda) &= P_\beta(s_\Lambda) + U_\beta(s_\Lambda)
\end{aligned}
\tag{3.1}
$$

For $\Lambda \in \mathscr{C}$, let $\mathfrak{B}(\Omega^\beta, \mathscr{F}_\Lambda)$ be the family of bounded functions depending only on the configurations $s_\Lambda \in (S^\beta)^\Lambda$. The local Gibbs measure on $(\Omega^\beta, \mathscr{F}_\Lambda)$ with zero boundary conditions is given by

$$\nu^{(0)}_{\Lambda,\beta}(f) = \int f(s_\Lambda)\, \nu^{(0)}_{\Lambda,\beta}(ds_\Lambda)$$

$$\equiv Z^{-1}_{\Lambda,\beta} \int \lambda_\beta(ds_\Lambda) \exp[-V_\beta(s_\Lambda)]\, f(s_\Lambda) \tag{3.2}$$

$$Z_{\Lambda,\beta} = \int \lambda_\beta(ds_\Lambda) \exp[-V_\beta(s_\Lambda)]$$

For $\Delta \subset \Lambda \in \mathscr{C}$ and $f \in (\Omega^\beta, \mathscr{F}_\Delta)$ the above can be expressed as

$$\nu^{(0)}_{\Lambda,\beta}(f) = Z^{-1}_{\Lambda,\beta} \int \lambda_\beta(ds_\Lambda) f(s_\Delta) \exp\left[-V_\beta(s_\Delta) - \sum_{i \in \Lambda\setminus\Delta} P_\beta(s_i)\right]$$

$$\times \exp\left[-\sum_{\substack{\{i,j\}:\\ i\in\Delta,\, j\in\Lambda\setminus\Delta}} U_\beta(s_i - s_j) - \sum_{\substack{\{i,j\}:\\ i,j\in\Lambda\setminus\Delta}} U_\beta(s_i - s_j)\right] \tag{3.3}$$

Denote by $b = \{i, j\}$ any two-point set in $\mathbf{Z}^\nu$, which is called a *bond* in $\mathbf{Z}^\nu$. Let $\mathscr{B}(\Lambda)$ be the family of bonds in Λ:

$$\mathscr{B}(\Lambda) = \{b : b = \{i,j\} \subset \Lambda\} \tag{3.4}$$

For given $b \in \mathscr{B}(\Lambda)$ we write

$$U_\beta(s_b) = U_\beta(|s_i - s_j|), \qquad b = \{i,j\} \tag{3.5}$$

$$h_\beta(s_b) = \exp[-U_\beta(s_b)] - 1 \tag{3.6}$$

Using the above notation, we may write

$$\exp\left[-\sum_{\substack{\{i,j\}:\\ i\in\Delta,\, j\in\Lambda\setminus\Delta}} U_\beta(|s_i - s_j|) - \sum_{\substack{\{i,j\}:\\ i,j\in\Lambda\setminus\Delta}} U_\beta(|s_i - s_j|)\right]$$

$$= \exp\left[-\sum_{\substack{b\in\mathscr{B}(\Lambda):\\ b\notin\mathscr{B}(\Delta)}} U_\beta(s_b)\right] \tag{3.7}$$

For given finite family $\hat{\mathscr{B}}$ of bonds (a family of two-point sets) one has the following *decoupling identities*:

$$\exp\left[-\sum_{b\in\hat{\mathscr{B}}} U_\beta(s_b)\right] = \prod_{b\in\hat{\mathscr{B}}} [h_\beta(s_b)+1]$$
$$= \sum_{\phi\subseteq\mathscr{B}\subset\hat{\mathscr{B}}} \prod_{b\in\mathscr{B}} h_\beta(s_b) \tag{3.8}$$

where the term corresponding to $\mathscr{B}=\varnothing$ is defined to be 1. For $X\subset\mathbf{Z}^\nu$ and $i\notin X$ it follows that

$$\exp\left[-\sum_{j\in X} U_\beta(|s_i-s_j|)\right] = \exp\left[-\sum_{\substack{b\in\mathscr{B}(X\cup\{i\}):\\ i\in b}} U_\beta(s_b)\right]$$
$$=1+\sum_{\substack{\varnothing\neq\mathscr{B}\subset\mathscr{B}(X\cup\{i\}):\\ b\in\mathscr{B}\Rightarrow i\in b}} \prod_{b\in\mathscr{B}} h_\beta(s_b)$$

For $b=\{i,j\}$ we write $h_\beta(s_b)=h_\beta(s_i,s_j)$. Then from the above one has the following *recoupling identities*:

$$\sum_{\varnothing\neq Y\subset X} \prod_{j\in Y:j\neq i} h_\beta(s_i,s_j)=\exp\left[-\sum_{j\in X} U_\beta(|s_i-s_j|)\right]-1 \tag{3.9}$$

Using the decoupling identities (3.8), we obtain that

$$\exp\left[-\sum_{\substack{b\in\mathscr{B}(\Lambda):\\ b\notin\mathscr{B}(\Delta)}} U_\beta(s_b)\right] = \sum_{\substack{\varnothing\subseteq\mathscr{B}\subset\mathscr{B}(\Lambda):\\ b\in\mathscr{B}\Rightarrow b\notin\mathscr{B}(\Delta)}} \prod_{b\in\mathscr{B}} h_\beta(s_b) \tag{3.10}$$

A family $\mathscr{B}=\{X_1, X_2,..., X_n\}$ of subsets of $\mathbf{Z}^\nu$ is said to be *connected* if for any X_i, $X_j\in\mathscr{B}$ there exists $\{X_{i_1},..., X_{i_l}\}\subset\mathscr{B}$ such that $X_{i_j}\cap X_{i_{j+1}}\neq\varnothing$ $(j=1,..., l-1)$, and $X_i\cap X_{i_1}\neq\varnothing$ and $X_j\cap X_{i_l}\neq\varnothing$. In the expression (3.10) we decompose $\mathscr{B}$ into the disjoint union of connected families (of bonds):

$$\mathscr{B}=\mathscr{B}_1\cup\mathscr{B}_2\cup\cdots\cup\mathscr{B}_m,$$
$$\mathscr{B}_i\cap\mathscr{B}_j=\varnothing\ (i\neq j)\qquad \text{and each } \mathscr{B}_i \text{ is connected}$$

For given $\Delta\subset\Lambda$ there exist $l(\geqslant 0)$ components which we may assume $\mathscr{B}_1, \mathscr{B}_2,..., \mathscr{B}_l$ such that $\mathscr{B}_i\cup\{\Delta\}$ is connected for $i=1,..., l$. For the sake of brevity we employ the following terminology: For $\Delta\subset\Lambda$ and families $\mathscr{B}_1, \mathscr{B}_2,..., \mathscr{B}_l$ of bonds we say that

"$P(\{\mathscr{B}_1,..., \mathscr{B}_l\}; \Delta, \Lambda)$ holds" if $\mathscr{B}_i\subset\mathscr{B}(\Lambda)$ $(i=1,..., l)$, $\mathscr{B}_i\cap\mathscr{B}_j=\varnothing$ $(i=j)$, $\mathscr{B}_i\cup\{\Delta\}$ is connected $(i=1,..., l)$, and $b\in\mathscr{B}_i\Rightarrow b\notin\mathscr{B}(\Delta)$ (3.11)

Let $\{\mathscr{B}_1,\ldots,\mathscr{B}_l\}$ be the maximal family for which $P(\{\mathscr{B}_1,\ldots,\mathscr{B}_l\};\Delta,\Lambda)$ holds. Resumming over $\mathscr{B}'=\bigcup_{j=l+1}^{m}\mathscr{B}_j$, and using (3.8) and (3.10), we obtain

$$
\begin{aligned}
\exp\Bigg[-\sum_{b\in\mathscr{B}(\Lambda):b\notin\mathscr{B}(\Delta)} U_\beta(s_b)\Bigg]
&= \sum_{\substack{\varnothing\subseteq\{\mathscr{B}_1,\ldots,\mathscr{B}_l\}:\\ P(\{\mathscr{B}_1,\ldots,\mathscr{B}_l\};\Delta,\Lambda)\text{ holds}}} \;\sum_{\substack{\mathscr{B}'\subset\mathscr{B}(\Lambda\backslash\Delta):\\ \mathscr{B}'\cap(\bigcup_{i=1}^{l}\mathscr{B}_i)=\varnothing}} \;\prod_{b\in\mathscr{B}'\cup(\bigcup_{i=1}^{l}\mathscr{B}_i)} h_\beta(s_b)\\
&=\Bigg(\sum_{\substack{\varnothing\subseteq\{\mathscr{B}_1,\ldots,\mathscr{B}_l\}:\\ P(\{\mathscr{B}_1,\ldots,\mathscr{B}_l\};\Delta,\Lambda)\text{ holds}}} \prod_{i=1}^{l}\prod_{b\in\mathscr{B}_i} h_\beta(s_b)\Bigg)\\
&\quad\times\exp\Bigg[-\sum_{b\in\mathscr{B}(\Lambda\backslash\Delta\cup(\bigcup\mathscr{B}_i))} U_\beta(s_b)\Bigg]
\end{aligned}
\tag{3.12}
$$

Again, for the sake of typographic convenience we use the following terminology: for a given family $\{b_1,\ldots,b_n\}$ of bonds and subsets Δ, $X\subset\mathbf{Z}^\nu$ we say that

$$
\text{“}P(\{b_1,\ldots,b_n\};\Delta,X)\text{ holds” if }\bigcup_{i=1}^{n} b_i=X,\ b_i\notin\mathscr{B}(\Delta)\ (i=1,\ldots,n),
$$

$$
\text{and if }\{b_1,\ldots,b_n\}\cup\{\Delta\}\text{ is connected} \tag{3.13}
$$

Then it is easy to check that

$$
\sum_{\substack{\varnothing\subseteq\{\mathscr{B}_1,\ldots,\mathscr{B}_l\}:\\ P(\{\mathscr{B}_1,\ldots,\mathscr{B}_l\};\Delta,\Lambda)\text{ holds}}}\cdots=\sum_{\substack{\varnothing\subseteq X\subset\Lambda:\\ \Delta\cap X\neq\varnothing\ (X\neq\varnothing)\\ X\backslash\Delta\neq\varnothing\ (X\neq\varnothing)}}\;\sum_{\substack{\{b_1,\ldots,b_n\}\subset\mathscr{B}(X):\\ P(\{b_1,\ldots,b_n\};\Delta,X)\text{ holds}}}\cdots \tag{3.14}
$$

For given Δ, $X\in\mathscr{C}$, define

$$
\hat{K}_\beta(\Delta,X;s_{\Delta\cup X})=\Bigg\{\sum_{\substack{\{b_1,\ldots,b_n\}\subset\mathscr{B}(X):\\ P(\{b_1,\ldots,b_n\};\Delta,X)\text{ holds}}}\prod_{i=1}^{n}h_\beta(s_{b_i})\Bigg\}
$$

$$
\times\exp\Bigg[-V_\beta(s_\Delta)-\sum_{i\in X\backslash\Delta}P_\beta(s_i)\Bigg]
$$

$$
\text{for}\quad X\backslash\Delta\neq\varnothing \tag{3.15a}
$$

$$
\hat{K}_\beta(\Delta,X;s_\Delta)=\exp[-V_\beta(s_\Delta)]\qquad\text{for}\quad X\backslash\Delta=\varnothing \tag{3.15b}
$$

and for $f \in \mathfrak{B}(\Omega^\beta, \mathscr{F}_\Delta)$ define

$$\hat{K}_\beta(\Delta, X; f) = \int \lambda_\beta(ds_{\Delta \cup X}) f(s_\Delta)\, \hat{K}_\beta(\Delta, X; s_{\Delta \cup X}) \tag{3.16}$$

We now use (3.3), (3.7), (3.12), and (3.14)–(3.16) (in that order) to obtain

$$\nu^{(0)}_{\Lambda,\beta}(f) = \sum_{\substack{\varnothing \subseteq X \subset \Lambda: \\ \Delta \cap X \neq \varnothing\ (X \neq \varnothing) \\ X \backslash \Delta \neq \varnothing\ (X \neq \varnothing)}} \hat{K}_\beta(\Delta, X; f) \left[\frac{Z_{\Lambda \backslash X \cup \Delta, \beta}}{Z_{\Lambda,\beta}} \right] \tag{3.17}$$

We write

$$Z^{(0)}_\beta \equiv \int \lambda_\beta(ds) \exp[-P_\beta(s)]$$

$$= \mathrm{Tr}(\exp[-\beta(-\tfrac{1}{2}\Delta + P)]) \tag{3.18}$$

For $\Delta \subset \Lambda \subset \mathbf{Z}^\nu$ and $f \in \mathfrak{B}(\Omega^\beta, \mathscr{F}_\Delta)$, put

$$K_\beta(\Delta, X; s_{\Delta \cup X}) \equiv (Z^{(0)}_\beta)^{-|\Delta \cup X|}\, \hat{K}(\Delta, X; s_{\Delta \cup X}) \tag{3.19a}$$

$$K_\beta(\Delta, X; f) \equiv \int \lambda_\beta(ds_{\Delta \cup X}) f(s_\Delta)\, K_\beta(\Delta, X; s_{\Delta \cup X}) \tag{3.19b}$$

$$\tilde{Z}_{\Lambda,\beta} \equiv (Z^{(0)}_\beta)^{-|\Lambda|}\, Z_{\Lambda,\beta} \tag{3.19c}$$

$$g_{\Lambda,\beta}(X) \equiv \frac{\tilde{Z}_{\Lambda \backslash X, \beta}}{\tilde{Z}_{\Lambda,\beta}} \tag{3.19d}$$

Then (3.17) and (3.19) imply that for $\Delta \subset X \subset \Lambda$ and $f \in \mathfrak{B}(\Omega^\beta, \mathscr{F}_\Delta)$

$$\nu^{(0)}_{\Lambda,\beta}(f) = \sum_{\substack{\varnothing \subseteq X \subset \Lambda: \\ \Delta \cap X \neq \varnothing\ (X \neq \varnothing) \\ X \backslash \Delta \neq \varnothing\ (X \neq \varnothing)}} K_\beta(\Delta, X; f)\, g_{\Lambda,\beta}(\Delta \cup X) \tag{3.20}$$

The above is the *cluster expansion for zero boundary conditions.*

Remark. Contrary to the cluster expansions in refs. 23 and 25, we did not expand the factor $\exp[-U_\beta(s_\Delta)]$ in the above expansion. See (3.7) and (3.12). This will simplify the proof of the convergence of the expansion.

3.2. A Cluster Expansion for General Gibbs Measures

For given Gibbs measure $\nu_\beta \in \mathscr{G}(\Omega^\beta)$ we use the equilibrium conditions in Definition 2.1.2 to write that for any $\Delta \subset \Lambda \in \mathscr{C}$ and $f \in \mathfrak{B}(\Omega^\beta, \mathscr{F}_\Delta)$

$$\nu_\beta(f) = \int \nu_\beta(ds) f(s)$$

$$= \int \nu_\beta(d\bar{s}) \left\{ Z_{\Lambda,\beta}(\bar{s})^{-1} \int \lambda_\beta(ds_\Lambda) f(s_\Delta) \exp[-V_\beta(s_\Lambda) - W_\beta(s_\Lambda, \bar{s}_{\Lambda^c})] \right\} \tag{3.21}$$

In order to develop a cluster expansion we write

$$\exp[-V_\beta(s_\Lambda) - W_\beta(s_\Lambda, \bar{s}_{\Lambda^c})]$$
$$= \exp\left[-V_\beta(s_\Delta) - \sum_{i \in \Lambda \backslash \Delta} P_\beta(s_i) - U_\beta^{(1)}(\Delta, \Lambda; s_\Lambda \bar{s}_{\Lambda^c}) \right] \tag{3.22}$$

where

$$U_\beta^{(1)}(\Delta, \Lambda; s_\Lambda \bar{s}_{\Lambda^c}) \equiv \sum_{\substack{\{i,j\}: \\ i \in \Delta, j \in \Lambda \backslash \Delta}} U_\beta(s_i - s_j) + \sum_{\{i,j\} \subset \Lambda \backslash \Delta} U_\beta(s_i - s_j)$$
$$+ \sum_{\substack{\{i,j\}: \\ i \in \Lambda, j \in \Lambda^c}} U_\beta(s_i - \bar{s}_j) \tag{3.23}$$

We remark that[12]

$$\exp[-W_\beta(s_\Lambda, \bar{s}_{\Lambda^c})] = \lim_{\Lambda' \to \mathbf{Z}^\nu} \exp[-W_\beta(s_\Lambda, \bar{s}_{\Lambda' \backslash \Lambda})]$$

for any $\bar{s} \in \mathfrak{S}$. Following the procedure from (3.7) to (3.14) in Section 3.1 and using a method similar to that used there one can derive the following equality:

$$\exp[-U_\beta^{(1)}(\Delta, \Lambda; s_\Lambda \bar{s}_\Lambda)]$$
$$= \sum_{\substack{\varnothing \subseteq X \in \mathscr{C}: \\ \Delta \cap X \neq \varnothing \ (X \neq \varnothing) \\ X \backslash \Delta \neq \varnothing \ (X \neq \varnothing)}} \left(\sum_{\substack{\{b_1, \ldots, b_n\} \subset \mathscr{B}(X): \\ P(\{b_1, \ldots, b_n\}; \Delta, X) \text{ holds} \\ b_i \notin \mathscr{B}(\Lambda^c),\ i = 1, \ldots, n}} \prod_{j=1}^n h_\beta(s_{b_j}) \right)$$
$$\times \exp[-U_\beta^{(2)}(\Delta, X, \Lambda; s_{\Lambda \backslash (\Delta \cup X)} \bar{s}_{\Lambda^c})] \tag{3.24}$$

where

$$U_\beta^{(2)}(\Delta, X, \Lambda; s_{\Lambda \backslash (\Delta \cup X)} \bar{s}_{\Lambda^c})$$
$$\equiv \sum_{\{i,j\} \subset \Lambda \backslash (\Delta \cup X)} U_\beta(s_i - s_j) + \sum_{\substack{\{i,j\}: \\ i \in \Lambda \backslash (\Delta \cup X) \\ j \in \Lambda^c \backslash (\Delta \cup X)}} U_\beta(s_i - \bar{s}_j) \tag{3.25}$$

We define that for $X \in \mathscr{C}$

$$\tilde{g}(X) \equiv \int \nu_\beta(ds) \exp\left[\sum_{\{i,j\} \subset X} U_\beta(s_i - s_j) + W_\beta(s_X, s_{X^c})\right] \tag{3.26}$$

Using (3.8), (3.25), and the equilibrium conditions, it can be checked that for $\Delta \cup X \subset \Lambda$

$$\begin{aligned}(Z_\beta^{(0)})^{|\Delta \cup X|} &\int \nu_\beta(d\bar{s})\, Z_{\Lambda,\beta}(\bar{s})^{-1} \int \lambda_\beta(ds_{\Lambda\setminus(\Delta\cup X)}) \\ &\times \exp\left[-\sum_{i \in \Lambda\setminus(\Delta\cup X)} P_\beta(s_i)\right] \exp[-U_\beta^{(2)}(\Delta, X, \Lambda; s_{\Lambda\setminus(\Delta\cup X)}\bar{s}_{\Lambda^c})] \\ &= \tilde{g}(\Delta \cup X)\end{aligned} \tag{3.27}$$

In (3.24) we divide the sum into two parts:

$$\sum_{X \in \mathscr{C}} \cdots = \sum_{X \subset \Lambda} \cdots + \sum_{X : X \cap \Lambda^c \neq \varnothing} \cdots$$

We first consider the contribution from $X \subset \Lambda$. If one substitutes (3.24) into (3.22) and then (3.22) into (3.21) and uses (3.27), one may observe that the term corresponding to $X \subset \Lambda$ is exactly $K_\beta(\Delta, X; f)\, \tilde{g}(\Delta \cup X)$, where $K_\beta(\Delta, X; f)$ has been defined in (3.19a).

Next we consider the contribution from X with $X \cap \Lambda^c \neq \varnothing$. Observe that for X with $X \cap \Lambda^c \neq \varnothing$

$$\begin{aligned}-V_\beta(s_\Delta) &- \sum_{i \in \Lambda\setminus\Delta} P_\beta(s_i) - U_\beta^{(2)}(\Delta, X, \Lambda; s_{\Lambda\setminus(\Delta\cup X)}\bar{s}_{\Lambda^c}) \\ &= -V_\beta(s_\Lambda) - W_\beta(s_\Lambda, \bar{s}_{\Lambda^c}) + U_\beta^{(3)}(\Delta, X, \Lambda; s_\Lambda \bar{s}_{\Lambda^c})\end{aligned} \tag{3.28}$$

where

$$\begin{aligned}U_\beta^{(3)}&(\Delta, X, \Lambda; s_\Lambda \bar{s}_{\Lambda^c}) \\ &= \sum_{\{i,j\} \subset \Lambda \cap (X\setminus\Delta)} U_\beta(s_i - s_j) + W_\beta(s_\Delta, s_{\Lambda\setminus\Delta}) \\ &\quad + \sum_{\substack{i \in \Lambda \cap (X\setminus\Delta) \\ j \in \Lambda\setminus(X\cup\Delta)}} U_\beta(s_i - s_j) + \sum_{\substack{i \in \Lambda^c \cap X \\ j \in \Lambda}} U_\beta(\bar{s}_i - s_j) \\ &\quad + \sum_{\substack{i \in \Lambda^c\setminus(X\cup\Delta) \\ j \in \Lambda \cap (X\cup\Delta)}} U_\beta(\bar{s}_i - s_j)\end{aligned} \tag{3.29}$$

We use (3.28) and the equilibrium conditions to conclude that for any $h \in L^1(\Omega^\beta, \lambda_\beta(ds_{\Delta \cup X}))$ and X with $X \cap \Lambda^c \neq \varnothing$

$$\int \nu_\beta(d\bar{s})\, Z_{\Lambda,\beta}(\bar{s})^{-1} \int \lambda(ds_\Lambda)\, h(s_{\Delta \cup X}) \exp\left[-V_\beta(s_\Delta) - \sum_{i \in \Lambda\setminus\Delta} P_\beta(s_i)\right]$$
$$\times \exp[-U_\beta^{(2)}(\Delta, X, \Lambda; s_{\Lambda\setminus(\Delta\cup X)}\bar{s}_{\Lambda^c})]$$
$$= \int \nu_\beta(ds)\, h(s_{\Delta \cup X}) \exp[U_\beta^{(3)}(\Delta, X, \Lambda; s_\Lambda \bar{s}_{\Lambda^c})] \tag{3.30}$$

Thus, combining the above results, we conclude that for any $f \in \mathfrak{B}(\Omega^\beta, \mathscr{F}_\Delta)$ and $\Delta \subset \Delta$

$$\nu_\beta(f) = \sum_{\varnothing \subseteq X \subset \Lambda} K_\beta(\Delta, X; f)\, \tilde{g}(\Delta \cup X) + R_{\Lambda,\beta}(f) \tag{3.31}$$

where

$$R_{\Lambda,\beta}(f) = \sum_{\substack{X \in \mathscr{C} \\ \Delta \cap X \neq \varnothing \\ \Lambda^c \cap X \neq \varnothing}} \int \nu_\beta(ds) f(s_\Delta) \sum_{\substack{\{b_1,\dots,b_n\} \subset \mathscr{B}(X) \\ P(\{b_1,\dots,b_n\};\Delta,X)\text{ holds} \\ b_i \notin \mathscr{B}(\Lambda^c)}} \prod_{j=1}^n h_\beta(s_{b_j})$$
$$\times \exp[U_\beta^{(3)}(\Delta, X, \Lambda; s_\Lambda s_{\Lambda^c})] \tag{3.32}$$

Here $U_\beta^{(3)}$ has been given in (3.29). The expression (3.31) is the *cluster expansion for general Gibbs measures.*

4. BASIC ESTIMATES

In this section we collect basic estimates which will be used in the proofs of the convergence of the cluster expansions in Section 3. We derive the estimates for quantum unbounded spin systems in Section 4.1. We also derive the necessary estimates for classical unbounded spin systems in Section 4.2 for the reader's convenience. Throughout this section we assume that the potential Φ satisfies the conditions on Assumption 2.2.1.

4.1. Basic Estimates: Quantum Systems

Recall the definition of $Z_\beta^{(0)}$ in (3.18). We use the following notation:

$$|s|_\beta = \left[\int_0^\beta |s(\tau)|^2\, d\tau\right]^{1/2} \tag{4.1}$$

for $s \in S^\beta$. Recall also the notations in (2.9) and (3.1).

Let $\gamma > 2$ be the constant in Assumption 2.2.1(b) and let α be a fixed positive number.

Lemma 4.1.1. For $0 < \beta \leqslant \alpha$ there exist positive constants c_1, c_2 and c_3 independent of β such that the following results hold:

(a) The bound

$$Z_\beta^{(0)} \geqslant c_1 \beta^{-d(1/2 + 1/\gamma)}$$

holds.

(b) For given $A \geqslant 0$ the bound

$$\int \lambda_\beta(ds) \exp[-P_\beta(s) + A\ |s|_\beta^2] \leqslant c_2 \beta^{-d(1/2 + 1/\gamma)}$$

holds.

(c) For given $A \geqslant 0$ the bound

$$\int \lambda_\beta(ds)\ |s|_\beta \exp[-P_\beta(s) + A\ |s|_\beta^2] \leqslant c_3 \beta^{-d(1/2 + 1/\gamma)} \beta^{(1 - 2/\gamma)/2}$$

holds.

The proof of the above lemma will be given at the end of this subsection. As a consequence one has the following results:

Corollary 4.1.2. For given $A \geqslant 0$ there exist positive constants c_1' and c_2' independent of β $(0 < \beta \leqslant \alpha)$ such that the bounds

$$(Z_\beta^{(0)})^{-1} \int \lambda_\beta(ds) \exp[-P_\beta(s) + A\ |s|_\beta^2] \leqslant c_1'$$

and

$$(Z_\beta^{(0)})^{-1} \int \lambda_\beta(ds)\ |s|_\beta \exp[-P_\beta(s) + A\ |s|_\beta^2] \leqslant c_2' \beta^{(1 - 2/\gamma)/2}$$

hold.

Proof. The corollary follows from Lemma 4.1.1. ∎

In the proof of the convergence of the cluster expansion (3.8) for general Gibbs measures, we will need β-dependent probability estimates for

Gibbs measures. Recall the function Ψ in Assumption 2.2.1(c). For given $\Delta \in \mathscr{C}$, $\nu_\beta \in \mathscr{G}^\Phi(\Omega^\beta)$, and constant $A \geqslant 0$, put

$$\nu_\beta(ds; A, \Delta) \equiv \exp\left[A \sum_{\{i,j\}: i \in \Delta, j \in \Delta^c} \Psi(|i-j|)\, |s_i|_\beta\, |s_j|_\beta\right] \nu_\beta(ds) \quad (4.2)$$

and let $\rho_\beta(s_\Delta; A, \Delta)$ be the distribution of $\nu_\beta(ds; A, \Delta)$ on $(\Omega_\Delta^\beta, \mathscr{F}_\Delta)$ with respect to $\lambda_\beta(ds_\Delta)$:

$$\nu_\beta(ds_\Delta; A, \Delta) = \rho_\beta(s_\Delta; A, \Delta)\, \lambda_\beta(ds_\Delta) \quad (4.3)$$

From the equilibrium conditions in Definition 2.1.2, it follows that for any $\Delta \subset \Lambda \subset \mathbf{Z}^\nu$

$$\rho_\beta(s_\Delta; A, \Delta) = \int \nu_\beta(d\bar{s})\, Z_{\Lambda,\beta}(\bar{s})^{-1} \int \lambda_\beta(ds_{\Lambda\setminus\Delta}) \exp[-V_\beta(s_\Lambda) - W_\beta(s_\Lambda, \bar{s}_{\Lambda^c})]$$
$$\times \exp\left[A \sum_{\{i,j\}: i \in \Delta, j \in \Delta^c} \Psi(|i-j|)\, |s_i|_b\, |\tilde{s}_j|_b\right] \quad (4.4)$$

where $\tilde{s}_j = s_j$ if $j \in \Lambda$, and $\tilde{s}_j = \bar{s}_j$ if $j \in \Lambda^c$. We also write that for $\Delta \subset \Lambda \in \mathscr{C}$

$$\rho_\beta^{(0)}(s_\Delta; A, \Delta) = Z_{\Lambda,\beta}^{-1} \int \lambda_\beta(ds_{\Lambda\setminus\Delta})$$
$$\times \exp\left[-V_\beta(s_\Lambda) + A \sum_{\{i,j\}: i \in \Delta, j \in \Lambda\setminus\Delta} \Psi(|i-j|)\, |s_i|_\beta\, |s_j|_\beta\right] \quad (4.5)$$

We then have the following results:

Proposition 4.1.3. Under the assumptions as in Lemma 4.1.1 one has the following results: (a) For any $\Delta \in \mathscr{C}$, $A \geqslant 0$, and $\nu_\beta \in \mathscr{G}^\Phi(\Omega^\beta)$, there exist constants $A^* > 0$ and $\delta > 0$ independent of β such that the bound

$$\rho_\beta(s_\Delta; A, \Delta) \leqslant \prod_{i \in \Delta} \beta^{d(1/2 + 1/\gamma)} \exp\left\{-\left[A^* \int_0^\beta |s_i(\tau)|^\gamma\, d\tau - \delta\right]\right\}$$

holds

(b) For any $\Delta \subset \Lambda \in \mathscr{C}$ and $A \geqslant 0$, there exist constants $\bar{A} > 0$ and $\bar{\delta} > 0$ independent of β such that the bound

$$\rho_{\Lambda,\beta}^{(0)}(s_\Delta; A, \Delta) \leqslant \prod_{i \in \Delta} \beta^{d(1/2 + 1/\gamma)} \exp\left\{-\left[\bar{A} \int_0^\beta |s_i(\tau)|^\gamma\, d\tau - \bar{\delta}\right]\right\}$$

holds.

The proof of the above proposition will be given in the Appendix. The above results are stronger than the Ruelle-type probability estimates in ref. 32.

Proof of Lemma 4.1.1. (a) For any $r>0$, put

$$\Sigma(r)=\{s\in S^\beta:\ \sup_{0\leqslant r\leqslant\beta}|s(\tau)|\leqslant r\} \tag{4.6}$$

Then one has that for $\beta>0$

$$\begin{aligned}
Z_\beta^{(0)} &\geqslant \int_{\Sigma(\beta^{-1/\gamma})} \lambda_\beta(ds)\exp[-P_\beta(s)]\\
&\geqslant c\int_{\Sigma(\beta^{-1/\gamma})} \lambda_\beta(ds)\exp\left[-D_3\int_0^\beta |s(\tau)|^\gamma\,d\tau\right]\\
&\geqslant c\int_{\Sigma(\beta^{-1/\gamma})} \lambda_\beta(ds)
\end{aligned}$$

Denote by $\Lambda(r)$ the largest box contained in the ball of radius r and let

$$\tilde{\Sigma}(r)=\{s\in S^\beta: s(\tau)\in\Lambda(r),\ \forall\tau\in[0,\beta]\}$$

Then from the above it follows that

$$\begin{aligned}
Z_\beta^{(0)} &\geqslant c\int_{\tilde{\Sigma}(\beta^{-1/\gamma})} \lambda_\beta(ds)\\
&= c\,\mathrm{Tr}\left(\exp\left[\frac{\beta}{2}\Delta_{\Lambda(\beta^{-1/\gamma})}\right]\right)\\
&\geqslant c_1\beta^{-d(1/2+1/\gamma)}
\end{aligned}$$

where $\Delta_{\Lambda(r)}$ is the Laplacian operator on $L^2(\Lambda(r), dx)$ with Dirichlet b.c.

(b) By Assumption 2.2.1(b) one obtains that the left-hand side of the inequality is bounded by

$$\begin{aligned}
\text{l.h.s.} &\leqslant c\int \lambda_\beta(ds)\exp\left[-\frac{D_3}{2}\int_0^\beta |s(\tau)|^\gamma\,d\tau\right]\\
&\leqslant c\,\mathrm{Tr}\left(\exp\left[-\beta\left(-\frac{1}{2}\Delta+\frac{D_3}{2}|x|^\gamma\right)\right]\right)\\
&\leqslant c(2\pi\beta)^{-d/2}\int_{\mathbf{R}^d} dx\exp\left(-\frac{1}{2}\beta D_3|x|^\gamma\right)\\
&\leqslant c_2\beta^{-d(1/2+1/\gamma)}
\end{aligned}$$

Here we have used the Golden–Thompson inequality[34] to get the third inequality.

(c) By the Hölder inequality one has that

$$|s|_\beta^2=\int_0^\beta |s(\tau)|^2\,d\tau\leqslant \beta^{1-(2/\gamma)}\left[\int_0^\beta |s(\tau)|^\gamma\,d\tau\right]^{2/\gamma} \tag{4.7}$$

Thus the left-hand side of the inequality is bounded by

$$\begin{aligned}\text{l.h.s.}&\leqslant \beta^{(1-2/\gamma)/2}\int \lambda_\beta(ds)\left[\int_0^\beta |s(\tau)|^\gamma\,d\tau\right]^{1/\gamma}\exp[-P_\beta(s)+A\,|s|_\beta^2]\\ &\leqslant \beta^{(1-2/\gamma)/2}\left[\int \lambda_\beta(ds)\left[\int_0^\beta |s(\tau)|^\gamma\,d\tau\right]\exp[-P_\beta(s)+A\,|s|_\beta^2]\right]^{1/\gamma}\\ &\quad\times\left[\int \lambda_\beta(ds)\exp[-P_\beta(s)+A\,|s|_\beta^2\right]^{1-1/\gamma}\end{aligned}$$

Here we have used the Hölder inequality to obtain the second inequality. Using the inequality $x\exp(-x)\leqslant c\exp(-x/2)$ for $x\geqslant 0$ and Assumption 2.2.1(b), we conclude that

$$\begin{aligned}\text{l.h.s.}&\leqslant c\beta^{(1-2/\gamma)/2}\int \lambda_\beta(ds)\exp\left[-\frac{1}{4}D_3\int_0^\beta |s(\tau)|^\gamma\,d\tau\right]\\ &\leqslant c_3\beta^{(1-2/\gamma)/2}\beta^{-d(1/2+1/\gamma)}\end{aligned}$$

Here we have used the method employed in the proof of part (b) of the lemma to get the last inequality. ∎

4.2. Basic Estimates: Classical Systems

We derive the basic estimates for the classical systems corresponding to Lemma 4.1.1–Proposition 4.1.3. Put

$$Z_\beta^{(0)}=\int dx\exp[-\beta P(x)] \tag{4.8}$$

We then have the following results:

Lemma 4.2.1. Let α be a fixed positive number. For $0<\beta\leqslant\alpha$ there exist positive constants c_1, c_2, and c_3 independent of $\beta>0$ such that the following results hold:

(a) The bound

$$Z_{\beta}^{(0)} \geqslant c_1 \beta^{-d/\gamma}$$

holds.

(b) For given $A \geqslant 0$ the bound

$$\int dx \exp[-\beta P(x) + A\beta x^2] \leqslant c_2 \beta^{-d/\gamma}$$

holds.

(c) For given $A \geqslant 0$ the bound

$$\int dx\, |\beta^{1/2} x| \exp[-\beta P(x) + A\beta x^2] \leqslant c_3 \beta^{-d/\gamma} \beta^{(1-2/\gamma)/2}$$

holds.

Proof. The lemma follows from (4.8), Assumption 2.2.1(b), and changes of variables. ▮

Corollary 4.2.2. For given $A \geqslant 0$ there exist positive constants c_1' and c_2' independent of β $(0 < \beta \leqslant \alpha)$ such that the bounds

$$(Z_{\beta}^{(0)})^{-1} \int dx \exp[-\beta P(x) + A\beta x^2] \leqslant c_1'$$

and

$$(Z_{\beta}^{(0)})^{-1} \int dx\, |\beta^{1/2} x| \exp[-\beta P(x) + A\beta x^2] \leqslant c_2' \beta^{(1-2/\gamma)/2}$$

hold.

Proof. The corollary follows from Lemma 4.2.1. ▮

Denote by $\mathscr{G}^{\beta\Phi}(\Omega)$ the space of Gibbs measures for classical systems,[19] where $\Omega = (\mathbf{R}^d)^{\mathbf{Z}^{\nu}}$. For given $\Delta \in \mathscr{C}$, $A \geqslant 0$, and $\nu_\beta \in \mathscr{G}^{\beta\Phi}(\Omega)$, put

$$\nu_\beta(d\omega; A, \Delta) \equiv \exp\left[A \sum_{\{i,j\} : i \in \Delta, j \in \Delta^c} \Psi(|i-j|)\, |x_i| \cdot |x_j| \right] \nu_\beta(d\omega)$$

and let $\rho_\beta(x_\Delta; A, \Delta)$ be the distribution of $\nu_\beta(d\omega; A, \Delta)$ on $(\Omega_\Delta, \mathscr{F}_\Delta)$ with respect to dx_Δ. We also write that for $\Delta \subset \Lambda \in \mathscr{C}$

$$\rho^{(0)}_{\Lambda,\beta}(x_\Delta; A, \Delta) = Z^{-1}_{\Lambda,\beta} \int dx_{\Lambda\setminus\Delta}$$

$$\times \exp\left[-\beta V(x_\Lambda) + A \sum_{\{i,j\}: i\in\Delta, j\in\Lambda\setminus\Delta} \Psi(|i-j|)\, |x_i| \cdot |x_j| \right]$$

where $Z_{\Lambda,\beta}$ is the partition function, i.e., the normalization factor for $A = 0$.

Proposition 4.2.3. Under the assumption as in Lemma 4.2.1 one has the following results:

(a) For any $\Delta \in \mathscr{C}$, $A \geqslant 0$, and $\nu_\beta \in \mathscr{G}^{\beta\Phi}(\Omega)$, there exist constants A^* and δ independent of β such that the bound

$$\rho_\beta(x_\Delta; A, \Delta) \leqslant \prod_{i\in\Delta} \beta^{d/\gamma} \exp(-A^*\beta\, |x_i|^\gamma + \delta)$$

holds.

(b) For any $\Delta \subset \Lambda \in \mathscr{C}$ and $A \geqslant 0$, there exist constants $\bar{A} > 0$ and $\bar{d} > 0$ independent of β such that the bound

$$\rho^{(0)}_{\Lambda,\beta}(x_\Delta; A, \Delta) \leqslant \prod_{i\in\Delta} \beta^{d/\gamma} \exp(-\bar{A}\beta\, |x_i|^\gamma + \bar{\delta})$$

holds.

The proof of the above proposition will be given in the Appendix.

5. CONVERGENCE OF THE CLUSTER EXPANSION: ZERO BOUNDARY CONDITIONS

We first need the following result:

Proposition 5.1. Let $\hat{\alpha}$ be a fixed positive number. For $0 < \beta \leqslant \hat{\alpha}$ there exists a constant $c > 0$ independent of β and $X \subset \Lambda \in \mathscr{C}$ such that the bound

$$g_\Lambda(X) \leqslant \exp(c\, |X|)$$

holds, where $|X| = \mathrm{card}(X)$.

Proof. From (3.19) it follows that

$$g_\Lambda(X) = (Z^{(0)}_\beta)^{|X|}\, Z_{\Lambda\setminus X,\beta} / Z_{\Lambda,\beta}$$

and

$$(Z_\beta^{(0)})^{|X|} Z_{\Lambda\backslash X,\beta} = \int \lambda_\beta(ds_\Lambda) \exp\Bigg[-V_\beta(s_\Lambda) + \sum_{\substack{\{i,j\} \subset \Lambda: \\ \{i,j\} \cap X \neq \varnothing}} U_\beta(s_i - s_j) \Bigg]$$

Notice that by Condition 2.2.1(c) there exists $A > 0$ independent of β such that

$$\begin{aligned} \sum_{\substack{\{i,j\} \subset \Lambda: \\ \{i,j\} \cap X \neq \varnothing}} U_\beta(s_i - s_j) &= \sum_{\{i,j\} \subset X} U_\beta(s_i - s_j) + W_\beta(s_X, s_{\Lambda\backslash X}) \\ &\leqslant A \sum_{i \in X} |s_i|_\beta^2 + \sum_{\substack{\{i,j\}: \\ i \in X, j \in \Lambda\backslash X}} \Psi(|i-j|)\, |s_i|_\beta\, |s_j|_\beta \end{aligned}$$

Thus we use (4.5) and Proposition 4.1.3(b) to conclude that

$$g_\Lambda(X) \leqslant \left\{ \beta^{d(1/2 + 1/\gamma)} \int \lambda_\beta(ds) \exp\left[-\bar{A} \int_0^\beta |s_i(\tau)|^\gamma \, d\tau + A\, |s_i|_\beta^2 + \bar{\delta} \right] \right.^{|X|}$$

Now the proposition follows from the method used in the proof of Lemma 4.1.1(b) and the above inequality. ▮

Let $\varepsilon > 0$ be the constant in Assumption 2.2.1(c). For given $c > 0$ and $\Delta \in \mathscr{C}$, put

$$\begin{aligned} A_\beta(\Delta) \equiv \sum_{\substack{\varnothing \neq X \in \mathscr{C}: \\ \Delta \cap X \neq \varnothing \\ X\backslash\Delta \neq \varnothing}} &\int \lambda(ds_{\Delta \cup X})\, |K_\beta(\Delta, X; s_{\Delta \cup X})| \\ &\times \exp\{c\, |\Delta \cup X| + \tfrac{1}{2}\varepsilon \log[\hat{d}(\Delta, X)]\} \end{aligned} \tag{5.1}$$

where $\hat{d}(\Delta, X) = \sup\{\operatorname{dist}(\Delta, i) : i \in X\}$. The main result in this section is the following:

Theorem 5.2. There exists $\beta_0 > 0$ such that for any $0 < \beta < \beta_0$ and $\Delta \in \mathscr{C}$ the series $A_\beta(\Delta)$ in (5.1) converges and the bound

$$A_\beta(\Delta) \leqslant e^{a|\Delta|} A(\beta)$$

holds, where a and $A(\beta)$ are constants such that $A(\beta) \to 0$ as $\beta \to \infty$.

We postpone the proof of the above theorem until later. As a consequence of the theorem we have the following result:

Theorem 5.3. There exists $\beta_0 > 0$ such that for $0 < \beta < \beta_0$ and $f \in \mathfrak{B}(\Omega^\beta, \mathscr{F}_\Delta)$ the bound

$$|\nu^{(0)}_{\Lambda,\beta}(f)| \leqslant \|f\|_\infty \, e^{a|\Delta|} A'(\beta)$$

holds uniformly in Λ, where $A'(\beta) \to 1$ as $\beta \to 0$.

Proof. Recall the cluster expansion in (3.20). By Proposition 5.1, Assumption 2.2.1(c), and Corollary 4.1.2 (in that order) we have

$$\sum_{\varnothing \subseteq X : X \subset \Delta} |K_\beta(\Delta, X; f)| \, g_\Lambda(\Delta \cup X)$$

$$\leqslant e^{c|\Delta|} (Z^{(0)}_\beta)^{-|\Delta|} \int \lambda_\beta(ds_\Delta) \, |f(s_\Delta)| \exp[-V_\beta(s_\Delta)]$$

$$\leqslant \|f\|_\infty \, e^{c|\Delta|} \left\{ (Z^{(0)}_\beta)^{-1} \int \lambda_\beta(ds_\Delta) \exp[-P_\beta(s) + A \, |s|^2_\beta] \right\}^{|\Delta|}$$

$$\leqslant \|f\|_\infty \, e^{a|\Delta|}$$

On the other hand, from (5.1), Proposition 5.1, and Theorem 5.2 it follows that

$$\sum_{\substack{\varnothing \neq X \subset \Lambda: \\ \Delta \cap X \neq \varnothing \\ X \setminus \Delta \neq \varnothing}} |K_\beta(\Delta, X; f)| \, g_\Lambda(\Delta \cup X) \leqslant \|f\|_\infty \, A_\beta(\Delta)$$

$$\leqslant \|f\|_\infty \, e^{a|\Delta|} A(\beta) \tag{5.2}$$

The theorem follows from the cluster expansion (3.20) and the above bounds. ∎

In the rest of this section we prove Theorem 5.2. Recall the definition of $\hat{K}_\beta(\Delta, X; s_{\Delta \cup X})$ in (3.15). We write $X = Y \cup W$ with $Y = \Delta \cap X$ and $W = X \setminus \Delta$. For Δ, Y, $W \in \mathscr{C}$ such that $Y \subset \Delta$ and $W \subset \mathbf{Z}^\nu \setminus \Delta$, put

$$I_\beta(\Delta; Y, W) \equiv \int \lambda_\beta(ds_\Delta) \, \lambda_\beta(ds_W) \, |\hat{K}_\beta(\Delta, Y \cup W; s_{\Delta \cup W})|$$

$$= \int \lambda_\beta(ds_\Delta) \, \lambda_\beta(ds_W) \exp\left[-V_\beta(s_\Delta) - \sum_{j \in W} P_\beta(s_j) \right]$$

$$\times \sum_{\substack{\{b_1, \ldots, b_n\} \subset \mathscr{B}(Y \cup W): \\ \bigcup b_j = Y \cup W, \, b_j \notin \mathscr{B}(Y), \\ \{b_1, \ldots, b_n\} \cup \{\Delta\} \text{ connected}}} \prod_{j=1}^n |h_\beta(s_{b_j})| \tag{5.3}$$

The right-hand side of (5.3) can be viewed as the sum over graphs $\{b_1, \ldots, b_n\}$ which are Δ-connected (i.e., $\{b_1, \ldots, b_n\} \cup \{\Delta\}$ is connected). We reduce the sum over Δ-connected graphs to that of Δ-connected tree graphs as follows:

Proposition 5.4. For given Δ, Y, $W=\{i_1,..., i_m\} \in \mathscr{C}$, the bound

$$I_\beta(\Delta; Y, \{i_1,..., i_m\}) \leqslant \int \lambda_\beta(ds_{\Delta \cup W}) \exp\left[-V_\beta(s_\Delta) - \sum_{j \in W} P_\beta(s_j)\right] \times \exp\left[\sum_{\substack{\{i,j\}: \\ i \in \{i_1,...,i_m\} \\ j \in Y \cup \{i_1,...,i_m\}}} |U_\beta(s_i - s_j)|\right] \prod_{k=1}^{m} \left[\sum_{j \in Y \cup \{i_1,...,i_{k-1}\}} |U_\beta(s_{i_k} - s_j)|\right]$$

holds.

Proof. For fixed Δ, Y, $W=\{i_1,..., i_m\}$ with $Y \subset \Delta$ and $W \subset \mathbf{Z}^\nu \backslash \Delta$, let $\{b_1,..., b_n\} \subset \mathscr{B}(Y \cup W)$ with $\bigcup b_j = Y \cup W$, $b_j \notin \mathscr{B}(Y)$, and $\{b_1,..., b_n\} \cup \{\Delta\}$ connected be given. Notice that there exists at least one $i_l \in W$ such that if one removes all the bonds adjacent to i_l (i.e., all $b=\{i_l, j\} \in \{b_1,..., b_n\}$) from $\{b_1,..., b_n\}$, what remains is still a Δ-connected family of bonds. By relabeling the elements of W one may assume $i_l = i_m$. Thus there exists $W' \subset Y \cup \{i_1,..., i_{m-1}\}$ such that

$$\prod_{j=1}^{n} |h_\beta(s_{b_j})| = \left[\prod_{\{i_m, i\}: i \in W'} |h_\beta(s_{i_m}, s_i)|\right]\left[\prod_{i=1}^{p} |h_\beta(s_{b_{j_i}})|\right] \tag{5.4}$$

where $\{b_{j_1},..., b_{j_p}\} \subset \{b_1,..., b_n\}$ is Δ-connected. We substitute (5.4) into (5.3) and perform the summation over $W' \subset Y \cup \{i_1,..., i_{m-1}\}$. By a recoupling identity (3.9) we have

$$\sum_{\varnothing \neq W' \subset Y \cup \{i_1,...,i_{m-1}\}} \prod_{\{i_m, i\}: i \in W'} |h_\beta(s_{i_m}, s_i)| \leqslant \exp\left[\sum_{j \in Y \cup \{i_1,...,i_{m-1}\}} |U_\beta(s_{i_m} - s_j)|\right] - 1 \tag{5.5}$$

This implies that for given $\varnothing \neq Y \subset \Delta$ and $\varnothing \neq W = \{i_1,..., i_m\} \subset \mathbf{Z}^\nu \backslash \Delta$ the bound

$$\sum_{\substack{\{b_1,...,b_n\} \subset \mathscr{B}(Y \cup W): \\ \bigcup b_j = Y \cup W,\ b_j \notin \mathscr{B}(Y), \\ \{b_1,...,b_n\} \cup \{\Delta\} \text{ connected}}} \prod_{j=1}^{n} |h_\beta(s_{b_j})| \leqslant \left\{\exp\left[\sum_{j \in Y \cup \{i_1,...,i_{m-1}\}} |U_\beta(s_{i_m} - s_j)|\right] - 1\right\} \times \sum_{\substack{\{b_1,...,b_p\} \subset \mathscr{B}(Y \cup \{i_1,...,i_{m-1}\}): \\ \bigcup b_j = Y \cup \{i_1,...,i_{m-1}\},\ b_j \notin \mathscr{B}(Y), \\ \{b_1,...,b_p\} \text{ connected}}} \prod_{i=1}^{p} |h_\beta(s_{b_i})| \tag{5.6}$$

Iterating the above inequality m times and using the fact that $e^x - 1 \leqslant |x|\, e^{|x|}$ for any $x \in \mathbf{R}$, the proposition follows from (5.3) and (5.6). ∎

We are now ready to prove Theorem 5.2.

Proof of Theorem 5.2. Notice that if $f(i_1,..., i_m)$ is a symmetric function on $\mathbf{Z}^\nu$, then

$$\sum_{\varnothing \neq X \subset \mathbf{Z}^\nu} f(X) = \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{i_1,...,i_m \in \mathbf{Z}^\nu}' f(i_1,...,i_m)$$

where $\sum'$ denotes the sum over $i_1,..., i_m \in \mathbf{Z}^\nu$ with the restriction that $i_k \neq i_l$ if $k \neq l$, k, $l = 1,..., m$. For given Δ, X with $\Delta \cap X \neq \varnothing$ and $X \backslash \Delta \neq \varnothing$, we write

$$X = Y \cup W, \qquad Y = X \cap \Delta, \qquad W = X \backslash \Delta$$

$$Y = \{j_1,...,j_q\}, \qquad W = \{i_1,...,i_m\}$$

From (5.1), (3.19a), and (5.3) it follows that

$$\begin{aligned} A_\beta(\Delta) = \sum_{q=1}^{|\Delta|} \sum_{m=1}^{\infty} \frac{1}{q!}\frac{1}{m!} \sum_{\substack{j_1,...,j_q \in \Delta \\ i_1,...,i_m \in \Delta^c}}' (Z_\beta^{(0)})^{-(|\Delta|+m)} \\ \times \exp\left\{c(|\Delta|+m) + \frac{\varepsilon}{2}\log\,[\hat{d}(\Delta, W)]\right\} \\ \times I_\beta(\Delta; \{j_1,...,j_q\}, \{i_1,...,i_m\}) \end{aligned} \tag{5.7}$$

where $\hat{d}(\Delta, W) = \sup\{\mathrm{dist}(\Delta, j) : j \in \{i_1,...,i_m\}\}$.

Next we estimate $I_\beta(\Delta; Y, W)$ by using Proposition 5.4. From Assumption 2.2.1(c) it follows that there exists $A > 0$ such that

$$\sum_{\{i,j\} \in \mathscr{B}(X)} |U_\beta(s_i - s_j)| \leqslant A \sum_{i \in X} |s_i|_\beta^2 \tag{5.8}$$

Write that for $\varnothing \neq Y \subset \Delta$ and $m \in \mathbf{N}$

$$\tilde{I}_\beta(\Delta; Y, m) \equiv \sum_{i_1,...,i_m \in \Delta^c}' \exp\left\{\frac{\varepsilon}{2}\log[\hat{d}(\Delta, W)]\right\} I_\beta(\Delta; Y, \{i_1,...,i_m\}) \tag{5.9}$$

where $W = \{i_1,...,i_m\}$. Using Proposition 5.4, the bound (5.8), and Assumption 2.2.1(c), we obtain that for $\varnothing \neq Y \subset \Delta$ and $\varnothing \neq W = \{i_1,...,i_m\}$

$$
\begin{aligned}
&I_\beta(\Delta;\, Y, \{i_1, ..., i_m\})\\
&\quad \leqslant \int \lambda_\beta(ds_\Delta) \exp\left[-V_\beta(s_\Delta) + A \sum_{j \in Y} |s_j|_\beta^2 \right]\\
&\quad \times \int \lambda_\beta(ds_W) \exp\left[-\sum_{i \in W} (P_\beta(s_i) - A\, |s_i|_\beta^2) \right]\\
&\quad \times \prod_{k=1}^{m} \left\{ \sum_{j \in Y \cup \{i_1, ..., i_{k-1}\}} \Psi(|i_k - j|)\, |s_{i_k}|_\beta\, |s_j|_\beta \right\}
\end{aligned} \tag{5.10}
$$

If one expands $\prod_{k=1}^{m} \{\cdots\}$, the right-hand side of (5.10) has $(|Y| + m - 1)!/|Y|!$ terms. One may recognize that each term can be labeled by a tree graph $T = T_1 \cup \cdots \cup T_l$ on the vertex set $Y \cup \{i_1, ..., i_m\}$ with some l $(1 \leqslant l \leqslant |Y|)$ connected components T_k, $k = 1, ..., l$. Notice that some of the T_i can be a singleton. Each connected component T_k has a root in Y. For each bond $\{i, j\} \in T$, we assign the factor $\Psi(|i-j|)\, |s_i|_\beta\, |s_j|_\beta$. In order to control the distance factor $\hat{d}(\Delta, W)$ in (5.9), we note that for any $x \geqslant 1$, $y \geqslant 1$, $x + y \leqslant 2xy$ and so by an induction the bound

$$
\log\left(\sum_{i=1}^{n} x_i \right) \leqslant \sum_{i=1}^{n} (\log x_i + \log 2)
$$

holds for any $x_i \geqslant 1$, $i = 1, ..., n$. By the above bound we conclude that

$$
\log[(\hat{d}(\Delta, W)] \leqslant \sum_{\{i,j\} \in T} [\log(|i-j|) + \log 2] \tag{5.11}
$$

for any tree graph $T = T_1 \cup \cdots \cup T_l$. Next we substitute (5.10) into (5.9) and use (5.11). Then instead of $\Psi(|i-j|)$ we have the factor $(2\,|i-j|)^{\varepsilon/2}\, \Psi(|i-j|)$ for each bond $\{i, j\} \in T$, which is summable by Assumption 2.2.1(c). Put

$$
D = 2 \sum_{i \in \mathbf{Z}^\nu} |i|^{\varepsilon/2}\, \Psi(|i|)
$$

We then perform the summations over $i_m \in \Delta^c, ..., i_1, \in \Delta^c$ and then resum over tree graphs to conclude that

$$
\begin{aligned}
\tilde{I}_\beta(\Delta;\, Y, m) \leqslant &\int \lambda_\beta(ds_\Delta) \exp\left[-V_\beta(s_\Delta) + A \sum_{j \in Y} |s_j|_\beta^2 \right]\\
&\times \int \prod_{k=1}^{m} \lambda_\beta(ds_k) \exp\left[-\sum_{k=1}^{m} (P_\beta(s_k) - A\, |s_k|_\beta^2) \right]\\
&\times \prod_{k=1}^{m} \left(\sum_{j \in Y \cup \{1, ..., k-1\}} D\, |s_k|_\beta\, |s_j|_\beta \right)
\end{aligned} \tag{5.12}
$$

Notice that the s_k are now just integral variables. Note that for any $A>0$

$$\prod_{k=1}^{m}\left(\sum_{j\in Y\cup\{1,\ldots,k-1\}} D\,|s_{i_k}|_\beta\,|s_j|_\beta\right)$$
$$\leqslant\left(\prod_{k=1}^{m} D\,|s_k|_\beta\right)\left(\sum_{j\in Y\cup\{1,\ldots,m\}} |s_j|_\beta\right)^m$$
$$\leqslant m!\,\bar{D}^m\left(\prod_{k=1}^{m} |s_k|_\beta\right)\exp\left(A\sum_{j\in Y\cup\{1,\ldots,m\}} |s_j|_\beta^2\right) \tag{5.13}$$

for some constant $\bar{D}$ independent of m and β. Thus from (5.12) and (5.13) we obtain the bound

$$\tilde{I}_\beta(\Delta;\,Y,m)\leqslant m!\,\bar{D}^m\int\lambda_\beta(ds_\Delta)\exp\left[-V_\beta(s_\Delta)+2A\sum_{j\in Y}|s_j|_\beta^2\right]$$
$$\times\left\{\int\lambda_\beta(ds)\,|s|_\beta\exp[-P_\beta(s)+2A\,|s|_\beta^2]\right\}^m \tag{5.14}$$

From (5.11), (5.8), (5.14), and Corollary 4.1.2 it follows that

$$A_\beta(\Delta)\leqslant\left\{\sum_{q=1}^{|\Delta|}\frac{1}{q!}(Z_\beta^{(0)})^{-|\Delta|}e^{c|\Delta|}\int\lambda_\beta(ds_\Delta)\right.$$
$$\left.\times\exp\left[-V_\beta(s_\Delta)+2A\sum_{j\in Y:|Y|=q}|s_j|_\beta^2\right]\right\}\sum_{m=1}^{\infty}(\tilde{c}\beta^{(1-2/\gamma)/2})^m$$
$$\leqslant A(\beta)\sum_{Y\subset\Delta}(Z_\beta^{(0)})^{-|\Delta|}e^{c|\Delta|}\int\lambda_\beta(ds_\Delta)$$
$$\times\exp\left[-V_\beta(s_\Delta)+2A\sum_{j\in Y}|s_j|_\beta^2\right]$$
$$=A(\beta)(Z_\beta^{(0)})^{-|\Delta|}e^{c|\Delta|}\int\lambda_\beta(ds_\Delta)$$
$$\times\left\{\prod_{i\in\Delta}[1+\exp(2A\,|s_i|_\beta^2)]\right\}\exp[-V_\beta(s_\Delta)]$$
$$\leqslant e^{a|\Delta|}A(\beta) \tag{5.15}$$

where $A(\beta)\to 0$ as $\beta\to 0$. This completes the proof of Theorem 5.2. ▮

6. UNIQUENESS AND CLUSTER PROPERTIES OF GIBBS STATES

6.1. Convergence of Cluster Expansion for General Gibbs Measures

We recall the cluster expansion (3.31) for general Gibbs measures. We state our results:

Proposition 6.1.1. Under Assumption 2.2.1 there exists $\beta_0 > 0$ such that for any $0 < \beta < \beta_0$ the following results hold:

(a) There exists a constant $c > 0$ such that

$$\tilde{g}(X) \leqslant \exp(c\,|X|)$$

(b) For any $f \in \mathfrak{B}(\Omega^\beta, \mathscr{F}_\Delta)$, $0 < \beta < \beta_0$, $R_{\Lambda,\beta}(f)$ is absolutely summable and

$$|R_{\Lambda,\beta}(f)| \leqslant \|f\|_\infty\, e^{c|\Delta|} \exp\left\{-\frac{\varepsilon}{2}\log[\operatorname{dist}(\Delta, \Lambda^c]\right\} A(\beta)$$

for some constant $c > 0$.

Corollary 6.1.2. For any $f \in \mathfrak{B}(\Omega^\beta, \mathscr{F}_\Delta)$, $0 < \beta < \beta_0$, and $\nu_\beta \in \mathscr{G}^\Phi(\Omega^\beta)$ one has

$$\nu_\beta(f) = \sum_{X \in \mathscr{C}: \Delta \cap X \neq \varnothing\, (X \neq \varnothing)} K_\beta(\Delta, X; f)\, \tilde{g}(\Delta \cup X)$$

The above expansion is absolutely summable.

Proof. The above result follows from (3.31), Proposition 6.1.1, and Proposition 5.2. ∎

Proof of Proposition 6.1.1. (a) Recall the definition of $\tilde{g}$ in (3.26). Since there exists a constant $A > 0$ such that

$$\begin{aligned} &\sum_{\{i,j\} \subset \mathscr{B}(X)} U_\beta(s_i - s_j) + W_\beta(s_X, s_{X^c}) \\ &\quad \leqslant J \sum_{i \in X} |s_i|_\beta^2 + \sum_{\{i,j\}: i \in X, j \in X^c} \Psi(|i-j|)\, |s_i|_\beta\, |s_j|_\beta \end{aligned}$$

it follows from (3.26) and Proposition 4.1.3(a) that

$$\tilde{g}(X) \leqslant \prod_{i\in X} \beta^{d(1/2+1/\gamma)} e^{\delta} \int \lambda_\beta(ds_i) \exp\left[-A^* \int_0^\beta |s_i(\tau)|^\gamma \, d\tau + J\,|s_i|_\beta^2 \right]$$

$$\leqslant \prod_{i\in X} \beta^{d(1/2+1/\gamma)} e^{D} \int \lambda_\beta(ds_i) \exp\left[-\bar{A} \int_0^\beta |s_i(\tau)|^\gamma \, d\tau \right]$$

$$\leqslant e^{c|X|}$$

Here we have used the method in the proof of Lemma 4.1.1(b) to get the last inequality.

(b) From (3.29) and Assumption 2.2.1(c) it follows that there exists a constant $J>0$ such that

$$U_\beta^{(3)} \leqslant 2J \sum_{i\in\Delta\cup X} |s_i|_\beta^2 + 2 \sum_{\substack{i\in\Delta\cup X\\ j\in(\Delta\cup X)^c}} \Psi(|i-j|)\,|s_i|_\beta\,|s_j|_\beta$$

Now instead of Δ and Λ in Proposition 4.1.3(a) we take $\Delta\cup X$ and $\Lambda\cup X$, respectively. Using Proposition 4.1.3(a), we obtain that

$$|R_{\Lambda,\beta}| \leqslant \sum_{\substack{X\in\mathscr{C}:\\ \Delta\cap X\neq\varnothing\\ \Lambda^c\cap X\neq\varnothing}} \int \lambda_\beta(ds_{\Delta\cup X}) \left\{ \sum_{\substack{\{b_1,\dots,b_n\}\subset\mathscr{B}(X):\\ P(\{b_1,\dots,b_n\};\Delta,X)\text{ holds}\\ b_i\notin\mathscr{B}(\Lambda^c)}} \prod_{i=1}^{n} |h_\beta(s_{b_i})| \right\}$$

$$\times \left\{ \prod_{i\in\Delta\cup X} \beta^{d(1/2+1/\gamma)} \exp\left[-A^* \int_0^\beta |s_i(\tau)|^\gamma \, d\tau + 2J\,|s_i|_\beta^2 + \delta \right] \right\}$$

By using the method in the proof of Lemma A.1.1(b)–(c), one obtains that for any $A^*>0$, $J>0$ there exist c_1' and c_2' independent of β such that

$$\int \lambda_\beta(ds) \exp\left[-A^* \int_0^\beta |s_i(\tau)|^\gamma \, d\tau + J\,|s_i|_\beta^2 \right] \leqslant c_1' \beta^{-d(1/2+1/\gamma)}$$

$$\int \lambda_\beta(ds)\,|s|_\beta \exp\left[-A^* \int_0^\beta |s_i(\tau)|^\gamma \, d\tau + J\,|s_i|_\beta^2 \right] \leqslant c_2' \beta^{(1-2/\gamma)/2} \beta^{-d(1/2+1/\gamma)}$$

One may follow the proof of Theorem 5.2 step by step to conclude that the bound in part (b) of the proposition holds. ∎

6.2. Kirkwood–Salsburg Type Integral Equations and Uniqueness

We first derive Kirkwood–Salsburg type integral equations[30] and then prove the uniqueness of Gibbs states by using the equations. Let f be a function defined on $\mathscr{C}$. Such functions form a Banach space $\mathfrak{F}_\xi$:

$$\mathfrak{F}_\xi = \{f\colon \|f\| = \sup_{X \in \mathscr{C}} \xi^{-|X|}\, |f(X)| < \infty\}, \qquad \xi > 0 \tag{6.1}$$

We propose to derive an equation of the form[30]

$$g_\Lambda = \mathbb{I} + K_\Lambda g_\Lambda$$

$$g = \mathbb{I} + Kg$$

where $\mathbb{I}(\varnothing) = 1$ and $\mathbb{I}(X) = 0$ if $|X| \neq 0$. Furthermore, we will show that $\|K_\Lambda\| < 1$, $\|K\| < 1$, and hence

$$g_\Lambda = (\mathbf{1} - K_\Lambda)^{-1}\, \mathbb{I}$$

$$g = (\mathbf{1} - K)^{-1}\, \mathbb{I}$$

are well defined and for any $X \in \mathscr{C}$ the limit

$$\lim_{\Lambda \to \mathbf{Z}^\nu} g_\Lambda(X) = g(X)$$

exists. In (3.15), we chose $\Delta = \{i_1\}$ for a fixed $i_1 \in \Lambda$. For $X \in \mathscr{C}$ with $i_1 \in X$, put

$$\hat{K}_\beta(\{i_1\}, X) \equiv \int \lambda_\beta(ds_X)\, \hat{K}_\beta(\{i_1\}, X; s_X) \tag{6.2}$$

Following the procedure in Section 3.1, it can be checked that

$$Z_{\Lambda \setminus (X - \{i_1\}), \beta} = Z_\beta^{(0)} Z_{\Lambda \setminus X, \beta} + \sum_{\substack{\varnothing \neq S \subset \Lambda \setminus (X - \{i_1\}) \\ i_1 \in S,\ |S| \geqslant 2}} \hat{K}_\beta(\{i_1\}, S)\, Z_{\Lambda \setminus (X \cup S), \beta} \tag{6.3}$$

As in Section 3.1, let

$$K_\beta(\{i_1\}, S) = \hat{K}_\beta(\{i_1\}, S) / (Z_\beta^{(0)})^{|S|} \tag{6.4}$$

For any β, we define an operator K_β on $\mathfrak{F}_\xi$ by

$$\begin{aligned} &(K_\beta f)(\varnothing) = 0 \\ &(K_\beta f)(X) = f(X - \{i_1\}) - \sum_{\substack{\varnothing \neq S \subset \mathbf{Z}^\nu \setminus (X - \{i_1\}) \\ i_1 \in S,\ |S| \geqslant 2}} \hat{K}_\beta(\{i_1\}, S) f(X \cup S) \end{aligned} \tag{6.5}$$

We introduce the operator χ_Λ on $\mathfrak{F}_\xi$ defined by

$$(\chi_\Lambda f)(X) = \chi_\Lambda(X) f(X) \tag{6.6}$$

where $\chi_\Lambda(X) = 1$ if $X \subset \Lambda$ and $\chi_\Lambda(X) = 0$ otherwise.

Lemma 6.2.1. Let g_Λ be given by (3.19d). Then the relation

$$g_\Lambda = \mathbb{I} + \chi_\Lambda K_\beta \chi_\Lambda g_\Lambda$$

holds for any $\Lambda \in \mathscr{C}$.

Proof. The lemma follows from (6.3) and (6.4). ▮

Proposition 6.2.2. There exists $\beta_0 > 0$ such that for any $0 < \beta < \beta_0$ the following results hold:

(a) For $\xi = e^c$ the bound

$$\|\chi_\Lambda K_\beta \chi_\Lambda\| < 1$$

holds uniformly in $\Lambda \in \mathscr{C}$, where c is the constant appearing in Theorem 5.1.

(b) The limit

$$g(X) = \lim_{\Lambda \to \mathbf{Z}^\nu} g_\Lambda(X)$$

exists for any $X \in \mathscr{C}$. Furthermore, the function g on $\mathscr{C}$ belongs to $\mathscr{F}_\xi$ and the equation

$$g = \mathbb{I} + K_\beta g$$

holds.

(c) For any $f \in \mathfrak{B}(\Omega^\beta, \mathscr{F}_\Delta)$ the infinite-volume limit

$$\nu_\beta^0(f) = \lim_{\Lambda \to \mathbf{Z}^\nu} \nu_{\Lambda,\beta}^0(f)$$

exists. Furthermore, the equation

$$\nu_\beta^0(f) = \sum_{X \in \mathscr{C} : \Delta \cap X \neq \varnothing \, (X \neq \varnothing)} K_\beta(\Delta, X; f)\, g(\Delta \cup X)$$

holds for any $f \in \mathfrak{B}(\Omega^\beta, \mathscr{F}_\Delta)$.

Proof. (a) From (6.5), (6.6), and Theorem 5.2 it follows that

$$\|\chi_\Lambda K_\beta \chi_\Lambda\| \leqslant e^{-c}\left(1 + \sup_{i_1 \in \mathbf{Z}^\nu} \sum_{i_1 \in S,\, |S| \geqslant 2} |K_\beta(\{i_1\}, S)|\, e^{c|S|}\right)$$
$$\leqslant e^{-c}[1 + e^a A(\beta)]$$

where $A(\beta) \to 0$ as $\beta \to 0$. Part (a) follows from the above bounds.

(b) As in the proof of (a), we can show that $\|K_\beta\| < 1$. Hence the equation

$$g = \mathbb{1} + K_\beta g$$

is well defined for a unique g. Using the standard argument in ref. 30, we can show that for any $X \in \mathscr{C}$, $g_\Lambda(X) \to g(X)$ as $\Lambda \to 0$. For the details we refer to ref. 30.

(c) This follows from (3.20), Theorem 5.2, and part (b) of the proposition. ∎

Now we are ready to prove the uniqueness of Gibbs states, Theorem 2.2.2. By the definition of Gibbs states we only need to show the uniqueness of Gibbs measures.

Proof of Theorem 2.2.2. Recall the definition of $\tilde{g}(X)$ in (3.26). By the equilibrium condition [see also (3.27)], we can write that for any $X \in \mathscr{C}(X \subset \Lambda)$

$$\tilde{g}(X) = \int \nu_\beta(d\bar{s})\, Z_{\Lambda,\beta}(\bar{s})^{-1} \int \lambda_\beta(ds_\Lambda)$$
$$\times \exp[-P_\beta(s_X) - V_\beta(s_{\Lambda\setminus X}) - W_\beta(s_{\Lambda\setminus X}, \bar{s}_{\Lambda^c})]$$

Adapting the method used in Section 3.2 [see also (6.3)], one obtains the following expansion:

$$\tilde{g}(X - \{i_1\}) = \tilde{g}(X) + \sum_{\substack{\varnothing \in S \subset \Lambda\setminus(X - \{i_1\}): \\ i_1 \in S,\, |S| \geqslant 2}} K_\beta(\{i_1\}, S)\, \tilde{g}(X \cup S)$$
$$+ \tilde{R}_{\Lambda,\beta}(X) \tag{6.7}$$

for any $i_1 \in X$, where $\tilde{R}_{\Lambda,\beta}(X)$ is the contribution from S with $i_1 \in S$ and $S \cap \Lambda^c \neq \varnothing$: The precise expression can be obtained from the expression of $R_{\Lambda,\beta}(f)$ in (3.32) by replacing Δ and X by $\{i_1\}$ and S, respectively, and by setting $f=1$ in (3.32). By the same reason as that in the proof of Proposition 6.1.1(b), it can be checked that

$$\lim_{\Lambda \to \mathbf{Z}^\nu} \tilde{R}_{\Lambda,\beta}(X)=0 \tag{6.8}$$

for any $X\in\mathscr{C}$, and so from (6.7), (6.8), and (6.5) one concludes that the function $\tilde{g}$ on $\mathscr{C}$ satisfies the integral equation

$$\tilde{g}=\mathbb{I}+K_\beta\,\tilde{g}$$

Since $\|K_\beta\|_\xi<1$ for any $0<\beta<\beta_0$ (by Lemma 6.2.1 and its proof), the above equation has a unique solution in $\mathfrak{F}_\xi$. By Proposition 6.2.2 we conclude that

$$g=\tilde{g}$$

Thus Corollary 6.1.2 and Proposition 6.2.2(c) imply that

$$\nu_\beta^{(0)}=\nu_\beta$$

for any $\nu_\beta\in\mathscr{G}^\Phi(\Omega^\beta)$. This completes the proof of the theorem. ∎

6.3. Cluster Properties: Proof of Theorem 2.2.3

We have developed the cluster expansion for Gibbs measure $\nu_\beta^{(0)}$ and the convergence of the cluster expansion for $0<\beta<\beta_0$. Let ν_β be the unique Gibbs measure. We then have the following cluster properties: for any $f_1\in\mathfrak{B}(\Omega^\beta,\mathscr{F}_{\Delta_1})$ and $f_2\in\mathfrak{B}(\Omega^\beta,\mathscr{F}_{\Delta_2})$

$$|\nu_\beta(f_1f_2)-\nu_\beta(f_1)\,\nu_\beta(f_2)|\to 0 \qquad \text{as} \quad \operatorname{dist}(\Delta_1,\Delta_2)\to\infty \tag{6.9}$$

Since there are well-known methods[6,7,30] to derive the cluster properties from the convergence of the cluster expansion, we will not produce the proof of (6.9) and refer the reader to the refs. 6, 7, and 30.

Next we consider the cluster properties of the unique Gibbs state $\rho\in\mathscr{G}_\beta^\Phi(\mathfrak{A})$ for $0<\beta<\beta_0$. For $\Delta\in\mathscr{C}$, let $\mathfrak{T}_\Delta^{(2)}$ be the class of Hilbert–Schmidt operators in $\mathfrak{A}_\Delta$. Since $\mathfrak{T}_\Delta^{(2)}$ is σ-weakly dense in $\mathfrak{A}_\Delta$ by the von Neumann density theorem, it suffices to show the cluster property for $A_1\in\mathfrak{T}_{\Delta_1}^{(2)}$ and $A_2\in\mathfrak{T}_{\Delta_2}^{(2)}$. For given $A\in\mathfrak{T}_\Delta^{(2)}$, let $h_A(x_\Delta,y_\Delta)$ be the integral kernel of A. Define

$$K_\beta(\Delta,X;A)\equiv(Z_\beta^{(0)})^{-|\Delta\cup X|}\int dx_\Delta\int dy_\Delta\,h_A(x_\Delta,y_\Delta)$$
$$\times\int P^\beta_{x_\Delta,y_\Delta}(d\hat{s}_\Delta)\int\lambda_\beta(ds_{X\backslash\Delta})\,\hat{K}_\beta(\Delta,X;\hat{s}_\Delta s_{X\backslash\Delta}) \tag{6.10}$$

One may compare the above expression to $K_\beta(\Delta, X; f)$ in (3.16) and (3.19). By the method in Section 3.1 one may derive the following cluster expansion:

$$\rho^{(0)}_{\Lambda,\beta}(A) = \sum_{\substack{\varnothing \subseteq X \subset \Lambda: \\ \Delta \cap X \neq \varnothing\ (X \neq \varnothing) \\ X \backslash \Delta \neq \varnothing\ (X \neq \varnothing)}} K_\beta(\Delta, X; A)\, g_{\Lambda,\beta}(\Delta \cup X) \tag{6.11}$$

for any $A \in \mathfrak{T}^{(2)}_\Delta$. We write that for $\hat{s} \in W_{x,y}$, $|\hat{s}|^2_\beta = \int_0^\beta |\hat{s}(\tau)|^2\, d\tau$. We remark that for any $c > 0$

$$\prod_{i \in \Delta} \int P^\beta_{x_i, y_i}(d\hat{s}_i) \exp[-P_\beta(\hat{s}_i) + c\, |\hat{s}_i|^2_\beta]$$
$$\leqslant c(\Delta, \beta) \prod_{i \in \Delta} \exp\left(-\frac{1}{2\beta} |x_i - y_i|^2\right)$$

and so a direct application of the method used in Section 5 proves the convergence of the expansion (6.11) for $A \in \mathfrak{T}^{(2)}_\Delta$. The cluster properties follow from the convergence of the cluster expansion. ∎

7. UNIQUENESS OF GIBBS STATES FOR ONE-DIMENSIONAL SYSTEMS

We consider quantum unbounded spin systems in $\mathbf{Z}$. In ref. 22 the systems are studied extensively and under an appropriate assumption on the pair potential it is proved that for any value $\beta > 0$ there exists an infinite-volume limiting Gibbs state which is translationally invariant and ergodic. Furthermore, it is analytic in terms of the self-interaction and two-body interaction potentials. The main tool used in ref. 22 is a polymer-type cluster expansion which differs from that in Section 4. Thus we do not know yet whether the state constructed in ref. 22 is a unique Gibbs state in the sense of Definition 2.1.2 (and Definition 2.1.4). In this section we prove that any one-dimensional system has a unique translationally invariant Gibbs state. That is, we produce the proof of Theorem 2.2.4. The main tool is a perturbation argument across the boundary ∂V of $A \in \mathscr{C}$.[2,15,29]

Consider one-dimensional systems and denote by $\mathscr{E}^\Phi(\Omega^\beta)$ the family of translationally invariant Gibbs measures on $(\Omega^\beta, \mathscr{F})$. By using the method in refs. 19 and 26 one may show that the measure constructed in ref. 22 is a Gibbs measure, and so $\mathscr{E}^\Phi(\Omega^\beta)$ is not empty. A straightforward application of the method in the proof of Theorem 2.7 in ref. 26 shows that $\mathscr{E}^\Phi(\Omega^\beta)$ is compact in the local convergence topology and a Choquet simplex.

For given $\Lambda \in \mathbf{Z}$, let $W_\beta(s_\Lambda, s_{\Lambda^c})$ be the interaction across the boundary of Λ defined in (2.12):

$$W_\beta(s_\Lambda, s_{\Lambda^c}) = \sum_{\{i,j\}\,:\, i\in\Lambda,\, j\in\Lambda^c} U_\beta(s_i - s_j)$$

From now on we suppress β in the notations if there is no confusion involved. For $\Lambda \in \mathscr{C}$, define a function $\tilde{W}_{\Lambda,\Lambda^c}$ on Ω by

$$\tilde{W}_{\Lambda,\Lambda^c}(s_\Lambda, s_{\Lambda^c}) = \sum_{i\in\Lambda} \sum_{j\in\Lambda^c} \Psi(|i-j|)\, |s_i| \cdot |s_j| \tag{7.1}$$

We then have the following result:

Lemma 7.1. Let $\Lambda = [-n, n]$ be an interval in $\mathbf{Z}$, and let the condition in Theorem 2.2.4 be satisfied. Then $\exp[\tilde{W}_{\Lambda,\Lambda^c}]$ is an element of $L^1(\Omega, d\nu)$ for any $\nu \in \mathscr{E}^{\Phi}(\Omega)$, and $\int \exp[\tilde{W}_{\Lambda,\Lambda^c}]\, d\nu$ is bounded uniformly in Λ.

Proof. For given $\Lambda = [-n, n]$, we write that

$$\tilde{W}^{(R)}_{\Lambda,\Lambda^c} = \sum_{|i|\leqslant n} \tilde{W}^{(R)}_{i,\Lambda^c}(s_i, s_{\Lambda^c})$$

$$\tilde{W}^{(R)}_{i,\Lambda^c}(s_i, s_{\Lambda^c}) = \sum_{j\geqslant n+1} \Psi(|i-j|)\, |s_i| \cdot |s_j|$$

and write $\tilde{W}^{(L)}_{\Lambda,\Lambda^c}$ analogously. For $\Lambda = [-n, n]$, put

$$p_i = D[\Psi(|n-i|]^{-1/2}$$

$$D = \sum_{|i|\leqslant n} [\Psi(|n-i|)]^{1/2}$$

Thus $p_i > 1$, $i \in \Lambda$, and $\sum_{|i|\leqslant n} p_i^{-1} = 1$. By the Hölder's inequality

$$\int \exp[\tilde{W}_{\Lambda,\Lambda^c}]\, d\nu \leqslant \left(\int \exp[2\tilde{W}^{R)}_{\Lambda,\Lambda^c}]\, d\nu\right)^{1/2} \left(\int \exp[2\tilde{W}^{(L)}_{\Lambda,\Lambda^c}]\, d\nu\right)^{1/2}$$
$$= \int \exp[2\tilde{W}^{(R)}_{\Lambda,\Lambda^c}]\, d\nu$$

and

$$\int \exp[2\tilde{W}^{(R)}_{\Lambda,\Lambda^c}]\, d\nu \leqslant \prod_{|i|\leqslant n} \left(\int \exp[2p_i \tilde{W}^{(R)}_{\{i\},\Lambda^c}]\, d\nu\right)^{1/p_i} \tag{7.2}$$

Notice that

$$2p_i \tilde{W}^{(R)}_{\{i\},\Lambda^c}(s_\Lambda, s_{\Lambda^c}) \leqslant 2D \sum_{j \geqslant n+1} [\Psi(|i-j|)]^{1/2} |s_i| \cdot |s_j|$$

Thus by Proposition 4.1.3(a), the right-hand side of (7.2) is bounded uniformly in Λ. This proves the lemma. ∎

For $\Lambda = [-n, n]$ and $m \in \mathbf{N}$, let

$$W^{(m)}(s_\Lambda, s_{\Lambda^c}) = \sum_{\substack{\{i,j\}: \\ i \in \Lambda, d(\Lambda^c, \{i\}) \leqslant m \\ j \in \Lambda^c, d(\Lambda^c, \{j\}) \leqslant m}} U(s_i - s_j) \tag{7.3}$$

Then for any $s \in \mathfrak{S}$

$$\lim_{m \to \infty} \exp[W^{(m)}(s_\Lambda, s_{\Lambda^c})] = \exp[W(s_\Lambda, s_{\Lambda^c})] \tag{7.4}$$

and

$$\exp[W^{(m)}(s_\Lambda, s_{\Lambda^c})] \leqslant \exp[\tilde{W}_{\Lambda,\Lambda^c}(s_\Lambda, s_{\Lambda^c})] \tag{7.5}$$

For $\Lambda = [-n, n]$, $m \in \mathbf{N}$, and $A \in \mathscr{F}_\Delta$ $(\Delta \subset \Lambda)$, we write

$$\begin{aligned} B_\Lambda(A; m) &\equiv \int \nu(ds)\, 1_A(s) \exp[W^{(m)}(s_\Lambda, s_{\Lambda^c})] \\ D_\Lambda(m) &\equiv \int \nu(ds) \exp[W^{(m)}(s_\Lambda, s_{\Lambda^c})] \end{aligned} \tag{7.6}$$

and

$$\nu_\Lambda^{(m)}(A) \equiv B_\Lambda(A; m)/D_\Lambda(m) \tag{7.7}$$

We also write

$$\begin{aligned} B_\Lambda(A) &\equiv \int \nu(ds)\, 1_A(s) \exp[W(s_\Lambda, s_{\Lambda^c})] \\ D_\Lambda &\equiv \int \nu(ds) \exp[W(s_\Lambda, s_{\Lambda^c})] \end{aligned} \tag{7.8}$$

and

$$\nu_\Lambda(A) \equiv B_\Lambda(A)/D_\Lambda \tag{7.9}$$

Lemma 7.2. For given $\Lambda = [-n, n]$, $\Delta \subset \Lambda$, and $A \in \mathscr{F}_\Delta$, the sequence $\{B_\Lambda(A; m)\}_{m \in \mathbf{N}}$ (resp. $\{D_\Lambda(m)\}_{m \in \mathbf{N}}$) converges to $B_\Lambda(A)$ (resp. D_Λ) uniformly in Λ.

Proof. Using the fundamental theorem of calculus and Assumption 2.2.1(c), one obtains

$$|B_A(A;m)-B_A(A)| \leqslant \int \nu(ds)|W(s_A,s_{A^c})-W^{(m)}(s_A,s_{A^c})| \times \exp[\tilde{W}_{A,A^c}(s_A,s_{A^c})] \qquad (7.10)$$

where $\tilde{W}_{A,A^c}$ has been defined in (7.1). We note that

$$|W(s_A,s_{A^c})-W^{(m)}(s_A,s_{A^c})| \leqslant \sum_{\substack{\{i,j\}:\\ i\in A,\, j\in A^c:\\ d(A^c,i)>m \text{ or } d(A,j)>m}} \Psi(|i-j|)\,|s_i|\cdot|s_j|$$

By Lemma 7.1

$$\int \nu(ds)\exp[\tilde{W}_{A,A^c}(s_A,s_{A^c})] \leqslant c$$

uniformly in A. Now the lemma for $\{B_A(A;m)\}$ follows from (7.10), the dominated convergence theorem, and the above bounds. The proof for $\{D_A(m)\}$ follows from the same argument. ▌

We are now ready to show Theorem 2.2.4.

Proof of Theorem 2.2.4. Let $\nu\in\mathscr{E}^{\Phi}(\Omega)$ be a fixed extremal element in $\mathscr{E}^{\Phi}(\Omega)$. By Assumption 2.2.1(c)

$$|W(s_A,s_{A^c})| \leqslant \tilde{W}_{A,A^c}(s_A,s_{A^c})$$

and so by Lemma 7.1, $\exp[W(s_A,s_{A^c})]$ is an element of $L^1(\Omega,d\nu)$ for any $A=[-n,n]$. By (7.9) and the equilibrium conditions one may check that

$$\nu_A=\nu_A^{(0)} \qquad \text{on} \quad (\Omega,\mathscr{F}_A) \qquad (7.11)$$

where $\nu_A^{(0)}$ is the local Gibbs state with zero boundary conditions. Let $\nu^{(0)}$ be the infinite-volume limit of $\nu_A^{(0)}$.[22] Then from (7.11) and Lemma 7.2 it follows that for any $A\in\mathscr{F}_\Delta$, $\Delta\in\mathscr{C}$,

$$\begin{aligned}
\nu^{(0)} &= \lim_{n\to\infty} \nu_{A_n}(A)\\
&= \lim_{n\to\infty}\,[\lim_{m\to\infty} \nu_{A_n}^{(m)}(A)]\\
&= \lim_{m\to\infty}\,[\lim_{n\to\infty} \nu_{A_n}^{(m)}(A)] \qquad (7.12)
\end{aligned}$$

where $\Lambda_n = [-n, n]$. We assert that for given $m \in \mathbb{N}$ and $A \in \mathscr{F}_\Delta$

$$\lim_{n \to \infty} \nu_{\Lambda_n}^{(m)}(A) = \nu(A) \tag{7.13}$$

We then conclude from (7.12) and (7.13) that

$$\nu^{(0)} = \nu$$

Since $\mathscr{E}^{\Phi}(\Omega)$ is a simplex, this proves the theorem.

We prove the assertion (7.13). Put

$$F_\Lambda^{(m)}(s_\Lambda, s_{\Lambda^c}) = \exp[W^{(m)}(s_\Lambda, s_{\Lambda^c})]/D_\Lambda(m)$$

Then $F_\Lambda^{(m)} \in L^2(\Omega, \nu)$ and

$$\nu_\Lambda^{(m)}(A) = \int \nu(ds)\, 1_A(s)\, F_\Lambda^{(m)}(s_\Lambda, s_{\Lambda^c})$$

By using (7.5), the method in the proof Lemma 7.1, and Jensen's inequality one can show that $\{\|F_{\Lambda_n}^{(m)}\|_2\}$ is bounded uniformly in Λ_n (and in m). Thus there exists a subsequence $\{F_{\Lambda_{n_j}}^{(m)}\}$ which converges weakly to an L^2-function, say $F_\infty^{(m)}$. Thus one has

$$\lim_{n_j \to \infty} \nu_{\Lambda_{n_j}}^{(m)}(A) = \int d\nu\, 1_A F_\infty^{(m)}$$

It is easy to check that $F_\infty^{(m)}$ is a $\mathscr{T}_\infty$-measurable function, where $\mathscr{T}_\infty$ is the algebra of tail events.[19] Since ν is trivial on $\mathscr{T}_\infty$ by the extremality of ν,[12] and $\nu(F_\infty^{(m)}) = 1$, we conclude that

$$\lim_{n_j \to \infty} \nu_{\Lambda_{n_j}}^{(m)}(A) = \nu(A)$$

We note that $\{n_\Lambda^{(m)}(A)\}$ is bounded uniformly in Λ. Since the above argument can be applied to any convergent subsequence of $\{n_{\Lambda_n}^{(m)}(A)\}_n$ the assertion is proved. ∎

APPENDIX. PROOFS OF PROPOSITIONS 4.1.3 AND 4.2.3

In this appendix we produce the proofs of explicitly β-dependent probability estimates in Propositions 4.1.3 and 4.2.3. We shall modify the probability estimates in refs. 26 and 32 in such a way that one extracts β dependences explicitly. The modifications are a β-dependent decomposition of the configuration space Ω^β and a β-independent λ substitution in refs. 26 and 32.

As in ref. 32, for given $\alpha > 0$ we can choose an integer $P_0 > 0$ and for each $n \geqslant P_0$ an integer $l_n > 0$ such that $|l_{n+1}/l_n - (1+2\alpha)| < 0$. Put

$$[n] = \{i \in \mathbf{Z}^\nu : |i| \leqslant l_n\}$$
$$V_n = (2l_n + 1)^\nu \tag{A.1}$$

The following is Proposition 2.1 of ref. 32.

Lemma A.1. Let $\varepsilon > 0$ and $c \geqslant 0$ be given, and let Ψ be the function on the natural integers given in Assumption 2.1.1(d) [also in Assumption 2.2.1(c)]. If α is sufficiently small, one can choose an increasing sequence $\{\psi_n\}$ such that $\psi_n \geqslant 1$, $\psi_n \to \infty$, and fix $P > P_0$ so that the following is true:

Let $n(\cdot)$ be a function from $\mathbf{Z}^\nu$ to the positive real numbers. Suppose that there exists q such that $q \geqslant P$ and q is the largest integer for which

$$\sum_{i \in [q]} n(i)^2 \geqslant \psi_q V_q$$

Then the bound

$$\sum_{i \in [q+1]} c + \sum_{i \in [q+1]} \sum_{j \notin [q+1]} \Psi(|i-j|)\, \tfrac{1}{2}[n(i)^2 + n(j)^2] \leqslant \varepsilon \sum_{i \in [q+1]} n(i)^2$$

holds.

Proof of Proposition 4.1.3. (a) Recall the notations in Eqs. (4.1)–(4.3). We first introduce a β-dependent decomposition of configuration space:

$$\mathfrak{R}_0 = \left\{ s \in \Omega^\beta : \frac{1}{\beta} \sum_{i \in [q]} \beta^{2/\gamma} |s_i|_\beta^2 \leqslant \psi_q V_q,\ \forall q \geqslant P \right\}$$

$$\mathfrak{R}_q = \left\{ s \in \Omega^\beta : \frac{1}{\beta} \sum_{i \in [q]} \beta^{2/\gamma} |s_i|_\beta^2 \leqslant \psi_q V_q \text{ and } \frac{1}{\beta} \sum_{i \in [l]} \beta^{2/\gamma} |s_i|_\beta^2 < \psi_l V_l,\right.$$
$$\left. \forall l \geqslant q+1 \right\} \tag{A.2}$$

$$\mathfrak{R} = \mathfrak{R}_0 \cup \left(\bigcup_{q \geqslant P} \mathfrak{R}_q \right)$$

From the definition of $\mathfrak{S}$ in (2.3) one has $\mathfrak{S} \subset \mathfrak{R}$. From the proof of Lemma 4.1.1(a) it follows that there exists a constant c independent of β such that the bound

$$\int_{\Sigma(\beta^{-1/\gamma})} \lambda_\beta(ds) \exp[-P_\beta(s)] \geqslant c\beta^{-d(1/2+1/\gamma)}$$

holds. Thus there exists β-independent constant $\lambda > 0$ such that the bound

$$1 \leqslant \lambda \beta^{d(1/2 + 1/\gamma)} \int_{\Sigma(\beta^{-1/\gamma})} \lambda_\beta(ds) \exp[-P_\beta(s)] \tag{A.3}$$

holds.

Recall the definition of $\rho_\beta(s_\Delta; A, \Delta)$ in (4.3). We write that

$$\rho_\beta(s_\Delta; A, \Delta) = \rho'_\beta(s_\Delta; A, \Delta) + \rho''_\beta(s_\Delta; A, \Delta)$$

where ρ'_β is the contribution from $\mathfrak{R}_0$ and ρ''_β the contribution from $\bigcup_{q \geqslant P} \mathfrak{R}_q$. We first consider ρ'_β. It follows from (4.4) that for any $\Delta \subset \Lambda$

$$\begin{aligned} &\rho'_\beta(s_\Delta; A, \Delta) \\ &\quad = \int \nu(d\bar{s}) \, Z_{\Lambda,\beta}(s)^{-1} \int \lambda_\beta(ds_{\Lambda\backslash\Delta}) \, 1_{\mathfrak{R}_0}(s_\Lambda \bar{s}_{\Lambda\backslash\Delta}) \\ &\qquad \times \exp\left[-V_\beta(s_\Lambda) - W_\beta(s_\Lambda, \bar{s}_{\Lambda\backslash\Delta}) + A \sum_{i \in \Delta, j \in \Delta^c} \Psi(|i-j|) \, |s_i|_\beta \, |\tilde{s}_j|_\beta \right] \end{aligned} \tag{A.4}$$

where $\tilde{s}_j = s_j$ if $j \in \Lambda$ and $\tilde{s}_j = \bar{s}_j$ if $j \in \Lambda^c$. Note that

$$\sum_{i \in [q]} |s_i|_\beta^2 \leqslant \beta^{1-2/\gamma} \psi_q V_q \qquad \text{on} \quad \mathfrak{R}_0$$

There exists a constant $D' > 0$ independent of β, $0 < \beta \leqslant \alpha$, such that for any $i \in \Delta$ and $s_\Lambda \bar{s}_{\Lambda^c} \in \mathfrak{R}_0$

$$\sum_{j \in \Delta^c} \Psi(|i-j|) \, |s_j|_\beta^2 \leqslant D' \tag{A.5}$$

See ref. 31 for the above bound. By Assumption 2.2.1 one has that for $k \in \Delta$

$$\begin{aligned} &-V_\beta(s_\Lambda) - W_\beta(s_\Lambda, \bar{s}_{\Lambda^c}) \\ &\quad = -P_\beta(s_k) - V_\beta(s_{\Lambda\backslash\{k\}}) - W_\beta(s_k, s_{\Lambda\backslash\{k\}}) - W_\beta(s_\Lambda, \bar{s}_{\Lambda^c}) \\ &\quad \leqslant -P_\beta(s_k) - V_\beta(s_{\Lambda\backslash\{k\}}) - W_\beta(s'_k, s_{\Lambda\backslash\{k\}}) - W_\beta(s'_k s_{\Lambda\backslash\{k\}}, \bar{s}_{\Lambda^c}) \\ &\qquad + J(|s_k|_\beta^2 + |s'_k|_\beta^2) + \sum_j \Psi(|k-j|) \, |\tilde{s}_j|_\beta^2 \end{aligned} \tag{A.6}$$

where $J = \sum_j \Psi(|j|)$. Thus from (A.5), (A.6), and the fact that $|xy| \leqslant (x^2 + y^2)/2$ it follows that for any $k \in \Delta$, $s_\Lambda \bar{s}_{\Lambda^c} \in \mathfrak{R}_0$

$$\begin{aligned}
&- V_\beta(s_\Lambda) - W_\beta(s_\Lambda, \bar{s}_{\Lambda^c}) + A \sum_{i \in \Delta, j \in \Delta^c} \Psi(|i-j|)\, |s_i|_\beta\, |\bar{s}_j|_\beta \\
&\quad \leqslant - V_\beta(s_{\Lambda \setminus \{k\}}) - W_\beta(s'_k, s_{\Lambda \setminus \{k\}}) - W_\beta(s'_k s_{\Lambda \setminus \{k\}}, \bar{s}_{\Lambda^c}) \\
&\qquad - P_\beta(s_k) + \tilde{J}(|s_k|_\beta^2 + |s'_k|_\beta^2) \\
&\qquad + A \sum_{i \in \Delta \setminus \{k\}, j \in \Delta^c} \Psi(|i-j|)\, |s_i|_\beta\, |\bar{s}_j|_\beta + D
\end{aligned} \tag{A.7}$$

for some constants $\tilde{J} > 0$ and $D > 0$ independent of β. Now we use (A.3), (A.4), and (A.7) to obtain that

$$\begin{aligned}
&\rho'_\beta(s_\Delta; A, D) \\
&\quad \leqslant \lambda \beta^{d(1/2 + 1/\gamma)} e^D \exp[-P_\beta(s_k) + \tilde{J}\, |s_k|_\beta^2] \sup_{s'_k \in \Sigma(\beta^{-1/\gamma})} \exp[\tilde{J}\, |s'_k|_\beta^2] \\
&\quad \times \int \nu(d\bar{s})\, Z_{\Lambda,\beta}(\bar{s})^{-1} \int_{\Sigma(\beta^{-1/\gamma})} \lambda_\beta(ds'_k) \int \lambda_\beta(ds_{\Lambda \setminus \{k\}}) \\
&\quad \times \exp\left[- V_\beta(s^*_\Lambda) - W_\beta(s^*_\Lambda, \bar{s}_{\Lambda^c}) + A \sum_{\substack{i \in \Delta \setminus \{k\} \\ j \in \Delta^c}} \Psi(|i-j|)\, |s_i|_\beta\, |\bar{s}_j|_\beta \right]
\end{aligned}$$

Notice that $|s'_k|_\beta^2 \leqslant \beta^{1-2/\gamma}$ on $\Sigma(\beta^{-1/\gamma})$. Thus from the above we conclude that the bound

$$\begin{aligned}
\rho'_\beta(s_\Delta; A, \Delta) \leqslant \beta^{d(1/2 + 1/\gamma)} \exp\left\{ - \left[A^* \int_0^\beta |s_k(\tau)|^\gamma\, d\tau - \delta \right] \right\} \\
\times \rho_\beta(s_{\Delta \setminus \{k\}}; A, \Delta \setminus \{k\})
\end{aligned} \tag{A.8}$$

holds for any $k \in \Delta$.

Next we consider ρ''_β. By the equilibrium conditions one may choose $\Lambda = \Lambda_q$ such that $\Delta \subset \Lambda_q$ and $[q+1] \subset \Lambda_q$ for each $q (\geqslant P)$, and such that

$$\begin{aligned}
\rho''_\beta(s_\Delta; A, \Delta) = \sum_{q \geqslant P} \int \nu(d\bar{s})\, Z_{\Lambda_q,\beta}(\bar{s})^{-1} \int \lambda_\beta(ds_{\Lambda_q \setminus \Delta})\, 1_{\mathfrak{R}_q}(s_{\Lambda_q} \bar{s}_{\Lambda_q^c}) \\
\times \exp\left[- V_\beta(s_{\Lambda_q}) - W_\beta(s_{\Lambda_q}, \bar{s}_{\Lambda_q^c}) \right. \\
\left. + A \sum_{\substack{i \in \Delta \\ j \in \Delta^c}} \Psi(|i-j|)\, |s_i|_\beta\, |\bar{s}_j|_\beta \right]
\end{aligned} \tag{A.9}$$

As in ref. 32, we have that for $[q+1] \subset V_q$

$$
\begin{aligned}
&-V_\beta(s_{\Lambda_q}) - W_\beta(s_{\Lambda_q}, \bar{s}_{\Lambda_q^c}) \\
&\quad = -V_\beta(s_{[q+1]}) - V_\beta(s_{\Lambda_q \setminus [q+1]}) - W_\beta(s_{[q+1]}, s_{\Lambda_q \setminus [q+1]}) \\
&\qquad - W_\beta(s_{[q+1]} s_{\Lambda_q \setminus [q+1]}, \bar{s}_{\Lambda_q^c}) \\
&\quad \leqslant -V_\beta(s_{[q+1]}) - V_\beta(s_{\Lambda_q \setminus [q+1]}) \\
&\qquad - W_\beta(s'_{[q+1]}, s_{\Lambda_q \setminus [q+1]}) - W_\beta(s'_{[q+1]} s_{\Lambda_q \setminus [q+1]}, \bar{s}_{\Lambda_q^c}) \\
&\qquad + \sum_{\substack{i \in [q+1] \\ j \in [q+1]^c}} \Psi(|i-j|)(|s_i|_\beta^2 + |\bar{s}_j|_\beta^2) \\
&\qquad + \sum_{\substack{i \in [q+1] \\ j \in [q+1]^c}} \Psi(|i-j|)(|s'_i|_\beta^2 + |\bar{s}_j|_\beta^2)
\end{aligned}
$$

Notice that there exist constants $\bar{A} > 0$ and $\delta > 0$ such that

$$
V_\beta(s_{[q+1]}) \geqslant \sum_{i \in [q+1]} \left(\bar{A} \int_0^\beta |s_i(\tau)|^\gamma \, d\tau - \delta \right)
$$

We now use Lemma A.1 and the above bounds to conclude that for $s_{\Lambda_q} \bar{s}_{\Lambda_q^c} \in \mathfrak{R}_q$

$$
\begin{aligned}
&-V_\beta(s_{\Lambda_q}) - W_\beta(s_{\Lambda_q}, \bar{s}_{\Lambda_q^c}) \\
&\quad \leqslant -\sum_{i \in [q+1]} \left\{ \left[\bar{A} \int_0^\beta |s_i(\tau)|^\gamma \, d\tau - \delta \right] - 2\varepsilon \beta^{-1+2/\gamma} |s_i|_\beta^2 + 2c \right\} \\
&\qquad - V_\beta(s_{\Lambda_q \setminus [q+1]}) - W_\beta(s'_{[q+1]}, s_{\Lambda_q \setminus [q+1]}) - W_\beta(s'_{[q+1]} s_{\Lambda_q \setminus [q+1]}, \bar{s}_{\Lambda_q^c}) \\
&\qquad + J \sum_{j \in [q+1]} |s'_j|_\beta^2
\end{aligned}
\tag{A.10}
$$

We add and subtract the factor $\varepsilon \sum_{i \in [q+1]} \beta^{-1+2/\gamma} |s_i|_\beta^2$ to (A.10), and then use (4.7) and (A.2) to conclude that for any fixed $\alpha > 0$ and $0 < \beta < \alpha$

$$
\begin{aligned}
&-V_\beta(s_{\Lambda_q}) - W_\beta(s_{\Lambda_q}, \bar{s}_{\Lambda_q^c}) \\
&\quad \leqslant -\sum_{i \in [q+1]} \left(A^* \int_0^\beta |s_i(\tau)|^\gamma \, d\tau - \bar{\delta} \right) - C'' \psi_{q+1} V_{q+1} \\
&\qquad - V_\beta(s_{\Lambda_q \setminus [q+1]}) - W_\beta(s'_{[q+1]}, s_{\Lambda_q \setminus [q+1]}) - W_\beta(s'_{[q+1]} s_{\Lambda_q \setminus [q+1]}, \bar{s}_{\Lambda_q^c}) \\
&\qquad + J \sum_{j \in [q+1]} |s'_j|_\beta^2
\end{aligned}
\tag{A.11}
$$

Notice that the constant C'' is independent of β. On the other hand, one may use Lemma A.1 to obtain

$$
\begin{aligned}
A \sum_{\substack{i \in \Delta \\ j \in \Delta^c}} \psi(|i-j|)\, |s_i|_\beta\, |s_j|_\beta
&\leqslant A \sum_{\substack{i \in [q+1] \\ j \in [q+1]^c}} \psi(|i-j|)\, |s_i|_\beta\, |s_j|_\beta + A \sum_{\substack{i \in [q+1] \\ j \in [q+1]}} \psi(|i-j|)\, |s_i|_\beta\, |s_j|_\beta \\
&\quad + A \sum_{\substack{i \in \Delta\backslash[q+1] \\ j \in \Delta^c}} \psi(|i-j|)\, |s_i|_\beta\, |s_j|_\beta \\
&\leqslant A\varepsilon\beta^{-(1-2/\gamma)} \sum_{i \in [q+1]} |s_i|_\beta^2 + AJ \sum_{i \in [q+1]} |s_i|_\beta^2 \\
&\quad + A \sum_{\substack{i \in \Delta\backslash[q+1] \\ j \in \Delta^c}} \psi(|i-j|)\, |s_i|_\beta\, |s_j|_\beta
\end{aligned}
\tag{A.12}
$$

Combining (A.11) and (A.12) and using (4.7) and the fact that $|s_j'|_\beta^2 \leqslant \beta^{1-2/\gamma}$ on $\Sigma(\beta^{-1/\gamma})$, one concludes that

$$
\begin{aligned}
&- V_\beta(s_{\Lambda_q}) - W_\beta(s_{\Lambda_q}, \bar{s}_{\Lambda_q^c}) + A \sum_{\substack{i \in \Delta \\ j \in \Delta^c}} \psi(|i-j|)\, |s_i|_\beta\, |s_j|_\beta \\
&\quad - \sum_{i \in [q+1]} \left(\tilde{A} \int_0^\beta |s_i(\tau)|^\gamma \, d\tau - \tilde{\delta} \right) \\
&\quad - V_\beta(s_{\Lambda_q\backslash[q+1]}) - W_\beta(s'_{[q+1]}, s_{\Lambda_q\backslash[q+1]}) - W_\beta(s'_{[q+1]} s_{\Lambda_q\backslash[q+1]}, \bar{s}_{\Lambda_q^c}) \\
&\quad + A \sum_{\substack{i \in \Delta\backslash[q+1] \\ j \in (\Delta\backslash[q+1])^c}} \psi(|i-j|)\, |s_i|_\beta\, |s_j|_\beta - C''\psi_{q+1} V_{q+1}
\end{aligned}
\tag{A.13}
$$

for some constants $\tilde{A}$, $\tilde{\delta}$, and $C'' > 0$ independent of β. We substitute (A.13) into (A.9), and then use (A.3) and the method employed in ref. 32 to obtain

$$
\begin{aligned}
&\rho''_\beta(s_\Delta; A, \Delta) \\
&\quad \leqslant \left(\prod_{i \in [q+1] \cap \Delta} \beta^{d(1/2 + 1/\gamma)} \right) \exp\left[- \sum_{i \in [q+1] \cap \Delta} \left(\tilde{A} \int_0^\beta |s_i(\tau)|^\gamma \, d\tau - \tilde{\delta} \right) \right] \\
&\quad \times \sum_{q \geqslant P} \exp(-C''\psi_{q+1} V_{q+1} + D'' V_{q+1})\, \rho_\beta(s_{\Delta\backslash[q+1]}; A, \Delta\backslash[q+1])
\end{aligned}
$$

As in ref. 32, part (a) of the proposition follows from (A.8), (A.14), and an induction on card(Δ).

(b) Throughout the proof of part (a) we fix Λ and set $\Lambda_q = \Lambda$ for all $q \geqslant P$ and then take the configuration $\bar{s}_{\Lambda^c}$ to be zero. Then the proof follows from that of part (a). ▌

Proof of Proposition 4.2.3. Replacing s_i by x_i, $i \in \Lambda$, and using the method employed in the proof of Proposition 4.1.3, the proposition follows from the method used in the quantum case. We leave the details to the reader. ▌

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